Vectorology

Vectors used extensively as tools to understand EM, not as objects to be studied in their own right. Thus, no proofs, but properties discussed to show how they are used. Homework will cover applications needed for EM.

Algebra

Cartesian components: $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$

Dot or scalar product: $\mathbf{A}\cdot\mathbf{B} = \left(A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}\right)\cdot\left(B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}\right)$

$$= A_x B_x + A_y B_y + A_z B_z$$

where $\mathbf{i}\cdot\mathbf{i} = 1 \qquad \mathbf{i}\cdot\mathbf{j} = 0 \qquad$ etc

Properties:

1) Let $\mathbf{B} = \mathbf{i}$. Then $\mathbf{A}\cdot\mathbf{i} = A_x$ etc

   In general, $\mathbf{A}\cdot\hat{\mathbf{b}}$ = projection of $\mathbf{A}$ along $\hat{\mathbf{b}}$

2) Let $\mathbf{B} = |\mathbf{B}|\,\mathbf{k} = B\,\mathbf{k}$. $\mathbf{A}\cdot\mathbf{k} = A_z$

   $$\cos\theta = \frac{A_z}{A} = \frac{\mathbf{A}\cdot\mathbf{k}}{A} = \frac{\mathbf{A}\cdot B\mathbf{k}}{A\,B} = \frac{\mathbf{A}\cdot\mathbf{B}}{A\,B} \qquad\qquad \mathbf{A}\cdot\mathbf{B} = A\,B\cos\theta$$

   Use to find the angle between two vectors.

Vector or cross product:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}\left(A_y B_z - A_z B_y\right) - \mathbf{j}\left(A_x B_z - A_z B_x\right) + \mathbf{k}\left(A_x B_y - A_y B_x\right)$$

Properties:

1) $\mathbf{A}\times\mathbf{B}$ is a vector

2) $\mathbf{A}\times\mathbf{B}$ is $\mathbf{A}$ and $\mathbf{B}$ direction determined by right hand rule

3) $\mathbf{A}\times\mathbf{B} = -\,\mathbf{B}\times\mathbf{A}$ from determinant form

4) Let $\mathbf{B} = B\,\mathbf{k}$ $\qquad\qquad$ $\mathbf{A} = A_y\,\mathbf{j} + A_z\,\mathbf{k}$

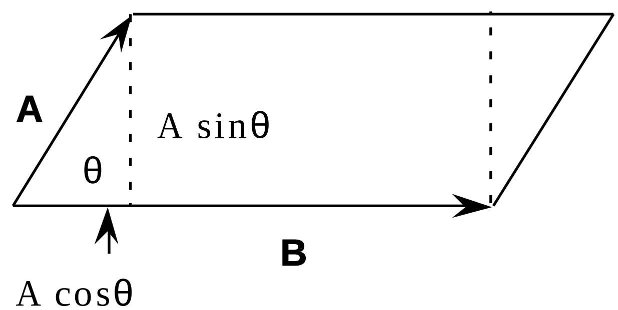
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & A_y & A_z \\ 0 & 0 & B \end{vmatrix} = \mathbf{i}\,A_y\,B - \mathbf{j}\,0 + \mathbf{k}\,0$$

ie, $\mathbf{A} \times \mathbf{B}$ $\quad \mathbf{A}, \mathbf{B}$

also, $\sin\theta = \dfrac{A_y}{A} = \dfrac{A_y\,B}{A\,B} = \dfrac{|\mathbf{A} \times \mathbf{B}|}{A\,B}$ $\qquad\qquad$ $\mathbf{A} \times \mathbf{B} = A\,B\,\sin\theta$

true in general

5) Geometry: area of plane parallelogram formed by $\mathbf{A}$, $\mathbf{B}$ is $|\mathbf{A} \times \mathbf{B}|$



$$\text{Area} = \frac{1}{2}\,A\cos\theta\,A\sin\theta + \left(B - A\cos\theta\right)A\sin\theta + \frac{1}{2}\,A\cos\theta\,A\sin\theta$$

$$= A^2\cos\theta\sin\theta + B\,A\sin\theta - A^2\cos\theta\sin\theta$$

$$= B\,A\sin\theta = |\mathbf{A} \times \mathbf{B}|$$

Scalar triple product:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ Cx & C_y & C_z \end{vmatrix}$$

Properties:

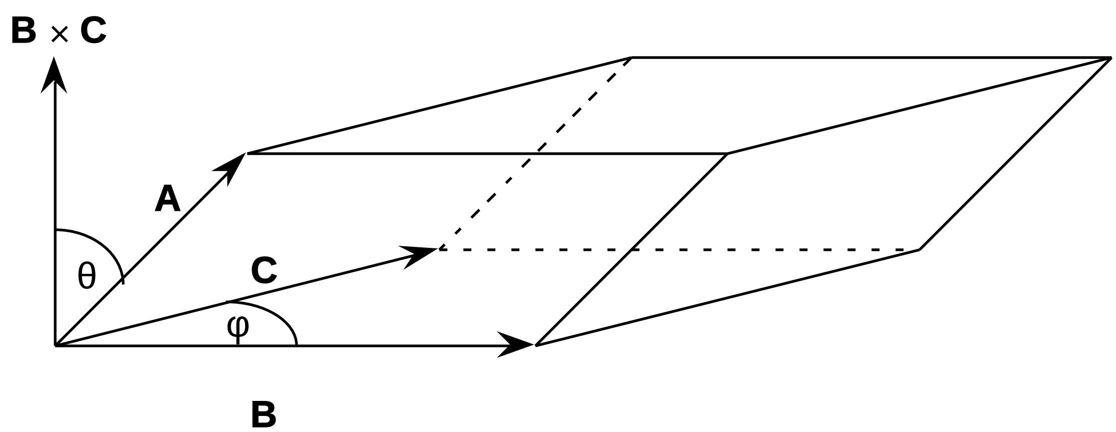
1) Order of operations is important: cross done first to give vector $\mathbf{A} \times \mathbf{B}$, then dot between two vectors. Other order does not make sense.

2) By direct calculation,

$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$ $\qquad$ permutation

$= \mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$ interchange $\cdot$ with $\times$

3) Geometry: $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ = volume of parallelopiped formed by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$



$\mathbf{B} \times \mathbf{C}$ = area of base

A $\cos\theta$ = height

vol = (area of base) (height) $= A \, |\mathbf{B} \times \mathbf{C}| \cos\theta = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$

$= A\,B\,C \sin\varphi \cos\theta$

Vector triple product:

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

Properties:

1) Order is important. ( ) are needed to specify order.

2) BAC CAB rule eliminates the $\times$'s

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}\,(\mathbf{A} \cdot \mathbf{B})$

Used often in EM.

Differentiation of scalar and vector functions

Define $\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$

$\nabla f(x, y, z)$ = gradient of f

Properties:

1) $\dfrac{\partial f}{\partial x}$ = gradient along x etc

∇f = vector whose components are the gradient along **i**, **j**, **k**

2) $\nabla f \cdot \mathbf{n} = |\nabla f| \cos\theta$ = gradient along **n** $|\mathbf{n}| = 1$

$\mathbf{n} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ reproduces definition of ∇f

3) **n** varied through θ, $\nabla f \cdot \mathbf{n} = |\nabla f| \cos\theta$ is max for $\theta = 0$

gradient is largest along its length

4) Used for 3D power series expansion:

$$f(\mathbf{r}) = f(\mathbf{r}_0) + \left.\frac{\partial f}{\partial x}\right|_{\mathbf{r}_0} \left(x - x_0\right) + \left.\frac{\partial f}{\partial y}\right|_{\mathbf{r}_0} \left(y - y_0\right) + \left.\frac{\partial f}{\partial z}\right|_{\mathbf{r}_0} \left(z - z_0\right) + \ldots$$

$$= f(\mathbf{r}_0) + \left.\nabla f\right|_{\mathbf{r}_0} \cdot \left(\mathbf{r} - \mathbf{r}_0\right) + \ldots$$

↑

rate of change of f in direction of $\mathbf{r} - \mathbf{r}_0$ times displacement

5) Allows converting scalar to vector: $\mathbf{E} = -\nabla\varphi$

Divergence:

Operate ∇ on a vector: $\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} = \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}$

Properties:

1) Converts a vector to a scalar, opposite of ∇

2) Interpretation: if **V** is the velocity of a fluid at each point **r**, $\nabla \cdot \mathbf{V}$ = rate at which fluid flows from the point **r**. $\nabla \cdot \mathbf{V}$ identifies the sources of flow.

Examples:

$\mathbf{V} = \mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z = \mathbf{r}$ **V** is a radial function

$\nabla \cdot \mathbf{V} = 1 + 1 + 1 = 3$ sources everywhere

$\mathbf{V} = a\,\mathbf{k}$ uniform flow along z

$\nabla \cdot \mathbf{V} = 0$ no sources anywhere

$\mathbf{V} = \mathbf{E}$ electric field

Maxwell equation $\nabla \cdot \mathbf{E} = 4\pi\rho$ source of electric field is charge density

3) terminology: $\nabla \cdot \mathbf{V} = 0$ $\mathbf{V}$ is "solenoidal"

Curl

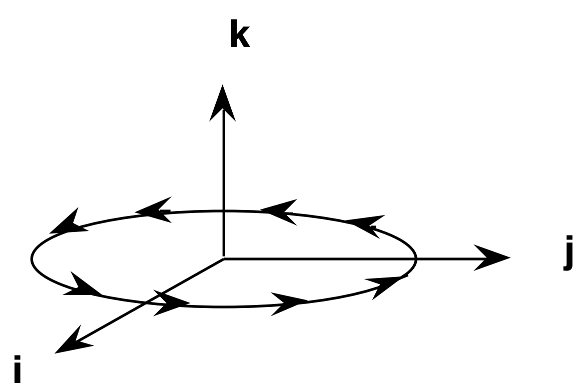
$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \mathbf{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Properties

1) curl takes vector to vector

2) Interpretation: selects component of velocity field that corresponds to rotation about a point

Examples: $\mathbf{V} = -y\,\mathbf{i} + x\,\mathbf{j}$



$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \mathbf{i}\; 0 - \mathbf{j}\; 0 + \mathbf{k}\,(1 + 1) = 2\;\mathbf{k}$$

direction of curl is axis of rotation.

$$\left.\begin{array}{l} \nabla \times (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 0 \\ \nabla \times a\,\mathbf{k} = 0 \end{array}\right\} \quad \text{no rotation}$$

$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$ Maxwell's equation $\qquad$ $\mathbf{B}$ curls around currents

3) terminology: $\nabla \times \mathbf{V} = 0$ $\qquad$ $\mathbf{V}$ is "irrotational"

∇ , $\nabla \cdot$, $\nabla \times$ are the spatial derivative operators for scalar and vector functions

Product rules:

derivative operators acting on products of functions, scalar-scalar, scalar-vector, vector-vector

$$\nabla (f\,g) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f\,g$$

$$= \mathbf{i}\left\{f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right\} + \mathbf{j}\left\{f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right\} + \mathbf{k}\left\{f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right\}$$

$$= f\,\nabla g + g\,\nabla f$$

All others derived same way, by direct differentiation:

$$\nabla \cdot (f\mathbf{A}) = f\,\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (f\,\mathbf{A}) = f\,(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla f$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B} + \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A})$$

$$\nabla\,(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathrm{A}) + (\mathbf{A} \cdot \nabla)\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{A}$$

Note meaning of $(\mathbf{B}\cdot\nabla)\,\mathbf{A} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)\mathbf{A}$

stands for 3 derivatives, one for each component of $\mathbf{A}$

$(\mathbf{B}\cdot\nabla)$ is a scalar operator, must act on scalar, each component of $\mathbf{A}$

These are tools to simplify vector calculus expressions, used often in EM

Our philosophy: take them from the list and use them.

2nd derivatives

five combinations of ∇ , $\nabla\cdot$, $\nabla\times$ applied successively

1) $\nabla\times(\nabla f) = 0$ prove by direct evaluation

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \mathbf{i}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right) \text{ etc } = 0 \quad \text{hinges on } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Thus, $\mathbf{A} = \nabla f$ automatically satisfies $\nabla\times\mathbf{A} = 0$

Will show that $\nabla\times\mathbf{E} = 0$ $\mathbf{E} = -\nabla\varphi$

2) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ also hinges on $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

3) $\nabla\cdot(\nabla f) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right)$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$= \nabla^2 f$ Laplacian operator

very important in EM

sometimes applied to vector:

$^{2}\mathbf{A}$ means vector whose components are $^{2}A_x$, $^{2}A_y$, $^{2}A_z$

4) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - {}^{2}\mathbf{A}$ prove by direct evaluation

often used in EM

5) $\nabla(\nabla \cdot \mathbf{A})$ occurs occasionally in EM, cannot be simplified further.

Usually eliminated with Maxwell equation, like $\nabla \cdot \mathbf{B} = 0$

Curvalinear coordinates

Cartesian: expressions for ∇ , $\nabla \cdot$, $\nabla \times$ are clear.

Other coordinate systems: more complicated, because the unit vector directions are functions of position; eg, $\hat{\mathbf{r}}$ varies with $\mathbf{r}$.

Spherical: $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\phi}$

etc look these up on the covers of Jackson, Griffiths

Integration of scalar and vector functions

model for all vector integrations is ordinary 1D line integration

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

for line integral, boundary is the two endpoints

scalar function integrated along a line in 3D:

$$\int_a^b \nabla f \cdot d\mathbf{l} = f(b) - f(a)$$ where line is arbitrary

for volume integral, boundary is surface

Divergence theorem

$$\int_V \nabla \cdot \mathbf{A} \; d^3r = \int_S \mathbf{A} \cdot \mathbf{n} \; da \qquad \mathbf{n} = \text{outward drawn normal}$$

Normally used in formal manipulations to get simple equations

Integration by parts

1 D:

$$\frac{d\,(fg)}{dx} = f\,\frac{dg}{dx} + g\,\frac{df}{dx}$$

$$\int_a^b \frac{d\,(fg)}{dx}\,dx = \int_a^b f\,\frac{dg}{dx}\,dx + \int_a^b g\,\frac{df}{dx}\,dx$$

$$\int_a^b f\,\frac{dg}{dx}\,dx = fg\Big|_a^b - \int_a^b g\,\frac{df}{dx}\,dx$$

characteristic features:

- differeentail moves to opposite side of integrand
- minus sign appears
- $fg\Big|_a^b$ is sometimes zero. This is usually the case if a, b $\rightarrow \infty$ and the

functions f, g $\rightarrow$ 0 at ∞.

In 3 D,

- volume integrals replace line integrals
- surface integrals replace evaluation at endpoints $fg\Big|_a^b$. Use the

divergence theorem to evaluate the derivative of the product.

- there are many product rules depending on the vector or scalar nature of f and g and the differential operation ∇ , $\nabla \cdot$, $\nabla \times$.

Example:

let $\mathbf{A} = f(\mathbf{r})\ \mathbf{B}(\mathbf{r})$ $\qquad \nabla \cdot \mathbf{A} = \nabla \cdot \left(f\ \mathbf{B}\right) = f\left(\nabla \cdot \mathbf{B}\right) + \mathbf{B} \cdot \nabla f$

$$\int_V \mathbf{B} \cdot \nabla f\ d^3r = \int_V \nabla \cdot \left(f\ \mathbf{B}\right) d^3r - \int_V f\left(\nabla \cdot \mathbf{B}\right) d^3r$$

$$= \int_S f\,\mathbf{B} \cdot \mathbf{n}\ da - \int_V f\left(\nabla \cdot \mathbf{B}\right) d^3r \qquad \text{divergence theorem}$$

take surface S at infinity, where physical quantities are zero.

$$= \quad - \int_V f\left(\nabla \cdot \mathbf{B}\right) d^3r$$

- differential operation moved to other factor in integrand
- minus sign appears (depending on product rule minus sign may not appear)

Corollary:

$$\int_V \nabla \times \mathbf{A}\ d^3r = \int_S \mathbf{n} \times \mathbf{A}\ da$$

prove by letting $\mathbf{A} = \mathbf{C} \times \mathbf{B}$ where $\mathbf{C}$ is a constant vector, in divergence theorem

Use for integration by parts with $\nabla \times$:

$$\int_V f(\nabla \times \mathbf{A})\ d^3r = \int_V \nabla \times \left(f\ \mathbf{A}\right) d^3r + \int_V \mathbf{A} \times \nabla f\ \ d^3r \quad \text{product rule}$$

$$= \int_S \mathbf{n} \times \left(f\ \mathbf{A}\right) da + \int_V \mathbf{A} \times \nabla f\ \ d^3r$$

↑ 0 if S at infinity

$$= \int_V \mathbf{A} \times \nabla f\ \ d^3r$$

- no minus sign for this product rule, due to order of $\mathbf{A} \times \nabla f$.

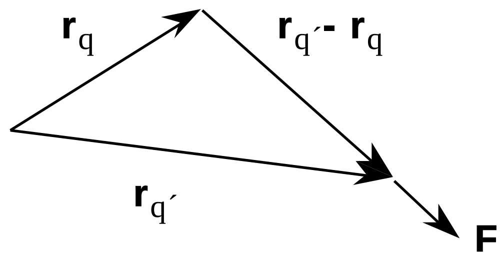
For surfaces bounded by contours:

Stokes theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{C is boundary for S}$$

Electrostatics

Coulomb's law



$$\mathbf{F} = \frac{qq'}{|\mathbf{r}_{q'} - \mathbf{r}_q|^2} \frac{\mathbf{r}_{q'} - \mathbf{r}_q}{|\mathbf{r}_{q'} - \mathbf{r}_q|} = \frac{qq'}{|\mathbf{r}_{q'} - \mathbf{r}_q|^3} (\mathbf{r}_{q'} - \mathbf{r}_q) \qquad \text{experimental law}$$

- magnitude is inverse square
- operates along line joining two charges
- beginning of quantitative study of electromagnetism, ~ 1795
- identical to gravitation same mathematical methods of potential theory
- despite similar form, fundamental differences:

Gravitation is much weaker than electrostatic forces

Gravitation does not interact with any other phenomena. It operates independent of temperature, pressure, electric or magnetic fields, light.

There is no "shield" for gravitation, whereas electrostatic force is interrupted, eg, by a grounded conductor.

- Einstein spent the last part of his life working on a single theory unifying electromagnetism and gravitation- unsuccessfully.

• Inverse square law was guessed by Priestly in 1767. Franklin noticed that cork balls inside a metal cup did not react to a charge on the cup, and asked Priestly to repeat the experiment. Priestly confirmed, and further speculated that the force law must be inverse square, in analogy to gravity, where it was known that a mass shell exerts no gravitational force inside it. Later, Priestly discovered oxygen (1771).

• Coulomb quantitatively established inverse square (1785) using a torsional balance which he invented. The analogy to gravitation gave this law and the idea of action at a distance a special place as the most complete statement of electrostatic phenomena.

Force law has two deficiencies:

1) It depends on magnitudes of two charges ("bi-linear in charge") making linear superposition complicated.

2) It emphasizes the physical charges rather than the space between the charges. Even in the absence of a second charge, the space surrounding a charge participates in electric effects- eg by propagating light.

Define electric field $$\mathbf{E}(\mathbf{r}) = \lim_{q'\to 0} \frac{\mathbf{F}}{q'} = \frac{q}{|\mathbf{r} - \mathbf{r}_q|^2} \frac{\mathbf{r} - \mathbf{r}_q}{|\mathbf{r} - \mathbf{r}_q|} = \frac{q}{|\mathbf{r} - \mathbf{r}_q|^3} (\mathbf{r} - \mathbf{r}_q)$$

Depends linearly on charge use superposition. One of the most useful and powerful concepts in EM.

Emphasizes a field in all space surrounding a charge. Fields have equal importance with charges in EM. Fields respond to charges, charges respond to fields.

• Idea of force acting at a distance with no intervening matter caused intellectual problems from the time of ancient Greece to Newton. They could not imagine a body moving unless it was physically touched. When they saw a compass needle orient, or the sun moving through the sky, they postulated an invisible medium which filled all space and moved in unseen ways to cause the observable motion in bodies. With Newton and inverse square laws which explained all the astronomical movements, the idea of action at a distance without an intervening medium became thinkable, though most workers, including Newton, felt there had to be a mechanism for transmitting the force which remained undiscovered.

For point charge at origin, $\nabla \times \mathbf{E} = 0$ verify by direct evalulation

$$\mathbf{E}(\mathbf{r}) = \frac{q}{r^2}\,\hat{\mathbf{r}}$$

spherical coordinates:
$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\varphi\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(r\, A_\varphi\right)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\, A_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}$$

$\mathbf{E}$ has only $\hat{\mathbf{r}}$ component, which depends only on r $\nabla \times \mathbf{E} = 0$

$\mathbf{E}$ can be written as gradient of scaler, $\mathbf{E} = -\nabla\varphi$

For point charge at origin at $\mathbf{r}'$

$$\varphi(\mathbf{r}) = \frac{q}{r} \qquad \frac{q}{|\mathbf{r} - \mathbf{r}'|}$$

prove by direct evaluation

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\hat{\phi}$$

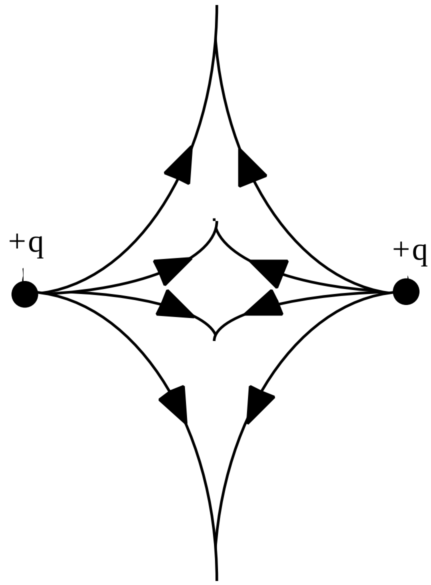
$$-\nabla\varphi = -\frac{\partial\varphi}{\partial r}\hat{\mathbf{r}} = \frac{q}{r^2}\hat{\mathbf{r}} = \mathbf{E}(\mathbf{r})$$

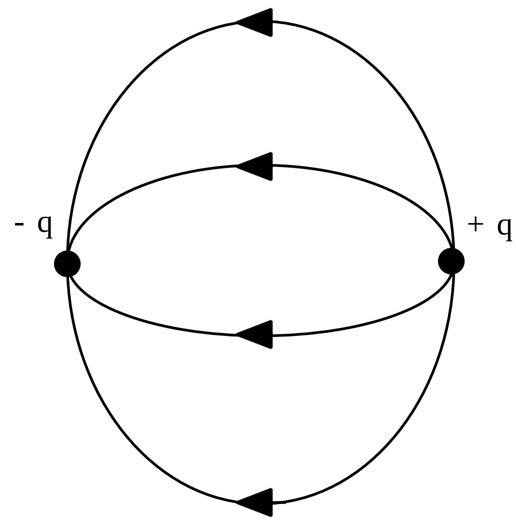
For many charges, use superposition: add vectorially, linearly

$$\mathbf{E}(\mathbf{r}) = \frac{q_1}{|\mathbf{r} - \mathbf{r}_1|^3}(\mathbf{r} - \mathbf{r}_1) + \frac{q_2}{|\mathbf{r} - \mathbf{r}_2|^3}(\mathbf{r} - \mathbf{r}_2) + \ldots$$

$$\varphi(\mathbf{r}) = \frac{q_1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{q_2}{|\mathbf{r} - \mathbf{r}_2|} + \ldots$$

Example: two point charges





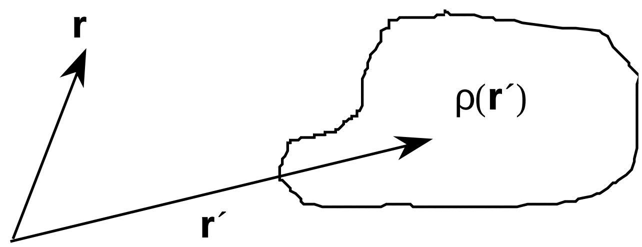
Continuous charge distribution

define $\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V}$ volume charge density $\frac{\text{charge}}{\text{unit vol}}$

$\sigma(\mathbf{r}) = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S}$ surface charge density $\frac{\text{charge}}{\text{unit area}}$

$\lambda(\mathbf{r}) = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l}$ line charge density $\frac{\text{charge}}{\text{unit length}}$

use superposition to write the field due to charge distribution- each volume element of charge acts independently



$$\mathbf{E}(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r' \qquad \rho(\mathbf{r}')\, d^3r' = \text{"point charge" at } \mathbf{r}'$$

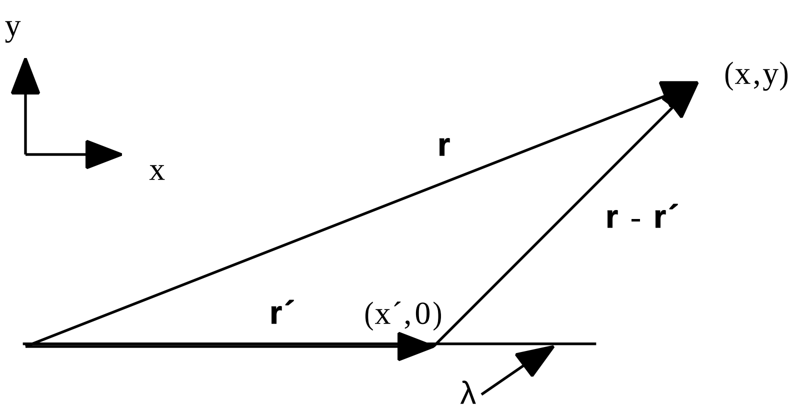
$$+ \int_S \frac{\sigma(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, d^2r' \qquad \sigma(\mathbf{r}')\, d^2r' = \text{"point charge" at } \mathbf{r}'$$

$$+ \int_C \frac{\lambda(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, dr' \qquad \lambda(\mathbf{r}')\, dr' = \text{"point charge" at } \mathbf{r}'$$

$$\varphi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' + \int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^2r' + \int_C \frac{\lambda(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dr'$$

Examples:

line of charge



$$\mathbf{E}(\mathbf{r}) = \int_C \frac{\lambda(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, dr'$$

Note: $\displaystyle \int \frac{1}{(a^2 + x^2)^{3/2}}\, dx = \frac{1}{a^2}\left(\frac{x}{(a^2 + x^2)^{1/2}}\right)$

$$\int \frac{x}{(a^2 + x^2)^{3/2}}\ dx = -\frac{1}{(a^2 + x^2)^{1/2}}$$

$$\int \frac{1}{(a^2 + x^2)^{1/2}}\ dx = \ln\left(x + (a^2 + x^2)^{1/2}\right)$$

$$E_x(\mathbf{r}) = \int_0^L \frac{\lambda\,(x - x')}{((x - x')^2 + y^2)^{3/2}}\ dx' = -\lambda \int_x^{x-L} \frac{u}{(u^2 + y^2)^{3/2}}\ du$$

change variables: $u = x - x'$, $du = -dx'$

$$= -\lambda \frac{1}{(y^2 + u^2)^{1/2}}\Bigg|_{x-L}^{x} = \frac{\lambda}{(y^2 + (x - L)^2)^{1/2}} - \frac{\lambda}{(y^2 + x^2)^{1/2}}$$

$$E_y(\mathbf{r}) = \int_0^L \frac{\lambda\, y}{((x - x')^2 + y^2)^{3/2}}\ dx' = -\lambda\, y \int_x^{x-L} \frac{1}{(u^2 + y^2)^{3/2}}\ du$$

change variables: $u = x - x'$, $du = -dx'$

$$= \lambda\, y \frac{1}{y^2}\left(\frac{u}{(y^2 + u^2)^{1/2}}\right)\Bigg|_{x-L}^{x} = \frac{\lambda}{y}\left\{\frac{x}{(x^2 + y^2)^{1/2}} - \frac{x - L}{((x - L)^2 + y^2)^{1/2}}\right\}$$

Limits: $x = L/2$

$$E_x = \frac{\lambda}{(y^2 + (-L/2)^2)^{1/2}} - \frac{\lambda}{(y^2 + (L/2)^2)^{1/2}} = 0$$

$$E_y = \frac{\lambda}{y}\left\{\frac{L/2}{((L/2)^2 + y^2)^{1/2}} - \frac{-L/2}{((-L/2)^2 + y^2)^{1/2}}\right\} = \frac{\lambda}{y}\frac{L}{((L/2)^2 + y^2)^{1/2}}$$

$L \to \infty$ (long straight wire)

$$E_y = \frac{\lambda}{y}\frac{L}{L/2} = \frac{2\lambda}{y}$$

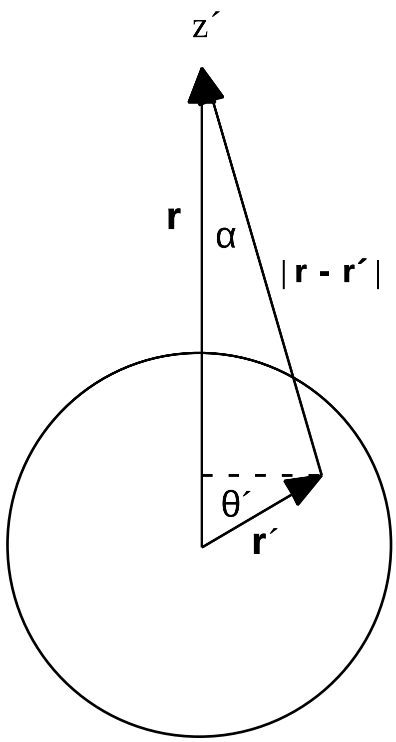
We will get this result later much more easily with Gauss's Law.

$$\varphi(\mathbf{r}) = \int_C \frac{\lambda(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\ dr' = \int_0^L \frac{\lambda}{((x - x')^2 + y^2)^{1/2}}\ dx'$$

$$= -\int_{L}^{x-L} \frac{\lambda}{(u^2 + y^2)^{1/2}}\, du = \lambda \ln\left(u + (y^2 + u^2)^{1/2}\right) \Big|_{x-L}^{x}$$

$$= \lambda \ln\left\{ \frac{x + (y^2 + x^2)^{1/2}}{x - L + (y^2 + (x - L)^2)^{1/2}} \right\}$$

Example 2: uniformly charged sphere



Choose z´ along **r**. Then

$$r' \cos\theta' + |\mathbf{r} - \mathbf{r}'| \cos\alpha = r$$
$$r' \sin\theta' = |\mathbf{r} - \mathbf{r}'| \sin\alpha$$

$$\begin{aligned}(r' \cos\theta' - r)^2 &= |\mathbf{r} - \mathbf{r}'|^2 \cos^2\alpha \\ &= |\mathbf{r} - \mathbf{r}'|^2 (1 - \sin^2\alpha) \\ &= |\mathbf{r} - \mathbf{r}'|^2 - r'^2 \sin^2\theta'\end{aligned}$$

$$r'^2 \cos^2\theta' - 2 r' r \cos\theta' + r^2 + r'^2 \sin^2\theta' = |\mathbf{r} - \mathbf{r}'|^2$$

$$r'^2 - 2 r' r \cos\theta' + r^2 = |\mathbf{r} - \mathbf{r}'|^2$$

“ law of cosines” from trigonometry

$$\varphi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' = \rho_0 \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

$$\int_V \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = \int_0^{2\pi} d\varphi \int_0^{a} r'^2\, dr' \int_0^{\pi} \sin\theta'\, d\theta' \frac{1}{\left(r^2 + r'^2 - 2\, r\, r' \cos\theta'\right)^{1/2}}$$

$$= 2\pi \int_0^a r'^2\, dr' \left\{ \left. \frac{\left(r^2 + r'^2 - 2rr'\cos\theta'\right)^{1/2}}{rr'} \right|_0^{\pi} \right\}$$

$$= 2\pi \int_0^a \frac{r'}{r} \left\{ \left(r^2 + r'^2 + 2rr'\right)^{1/2} - \left(r^2 + r'^2 - 2rr'\right)^{1/2} \right\} dr'$$

$$= 2\pi \int_0^a \frac{r'}{r} \left\{ \sqrt{(r + r')^2} - \sqrt{(r - r')^2} \right\} dr'$$

$$= 2\pi \int_0^a \frac{r'}{r} \left\{ |r + r'| - |r - r'| \right\} dr'$$

For $r > a$, $r > r'$ $\qquad |r - r'| = r - r'$

$$\int_V \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = \frac{4\pi}{r} \int_0^a r'^2\, dr' = \frac{4\pi}{3} a^3 \frac{1}{r}$$

$$\varphi(\mathbf{r}) = \rho_0 \frac{4\pi}{3} a^3 \frac{1}{r} = \frac{Q}{r} \qquad Q = \text{total charge}$$

Looks like a point charge at the origin.

For $r < a$, $\quad r > r'$ $\qquad |r - r'| = r - r'$

$\qquad r < r'$ $\qquad |r - r'| = r' - r$

$$\int_V \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = 2\pi \int_0^r \frac{r'}{r} 2r'\, dr' + 2\pi \int_r^a \frac{r'}{r} 2r\, dr'$$

$$= 4\pi\left\{\frac{r^3}{3r} + \frac{a^2 - r^2}{2}\right\} = 2\pi a^2 - \frac{2}{3}\pi r^2$$

$$\varphi(\mathbf{r}) = \rho_0\left(2\pi a^2 - \frac{2}{3}\pi r^2\right)$$

Note continuity at r = a :

$r = a^+$ $\qquad \varphi(\mathbf{r}) = \rho_0 \frac{4\pi}{3} a^3 \frac{1}{a} = \rho_0 \frac{4\pi}{3} a^2$

$r = a^-$ $\qquad \varphi(\mathbf{r}) = \rho_0\left(2\pi a^2 - \frac{2}{3}\pi a^2\right) = \rho_0 \frac{4\pi}{3} a^2$

If no continuity, $\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r})$ would be infinite. Physically impossible.

$$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r})$$

$r > a$ $\qquad \mathbf{E}(\mathbf{r}) = -\nabla\rho_0 \frac{4\pi}{3} a^3 \frac{1}{r} = \rho_0 \frac{4\pi}{3} a^3 \frac{1}{r^2}\,\hat{\mathbf{r}} = \frac{Q}{r^2}\,\hat{\mathbf{r}}$

looks like point charge Q at origin

$r < a$ $\qquad \mathbf{E}(\mathbf{r}) = -\nabla\rho_0\left(2\pi a^2 - \frac{2}{3}\pi r^2\right) = \rho_0 \frac{4}{3}\pi r\,\hat{\mathbf{r}} = \rho_0 \frac{4}{3}\pi\,\mathbf{r}$

Note continuity of E at r = a:

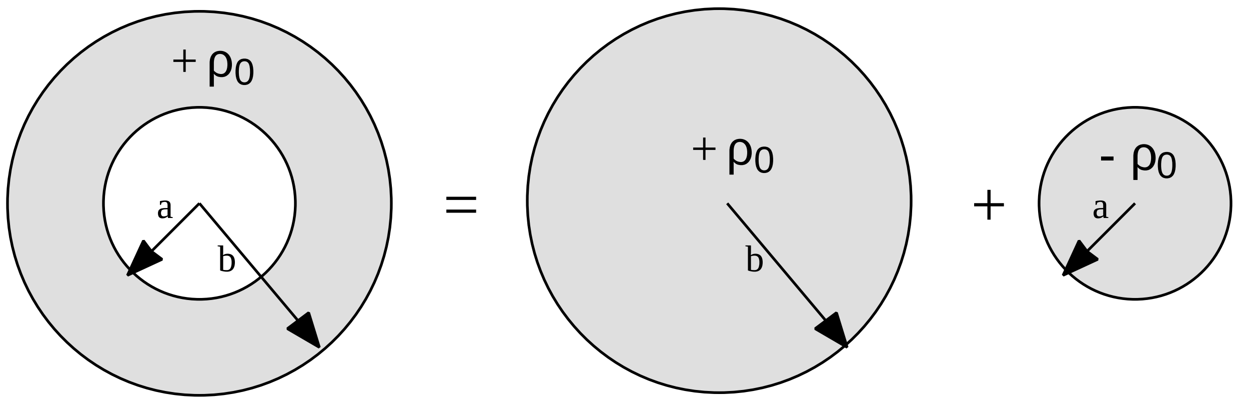
$r = a^+$ $\qquad E = \rho_0 \frac{4\pi}{3} a$

$r = a^-$ $\qquad E = \rho_0 \frac{4}{3}\pi a$

In general, **E** does not have to be continuous. We will find examples of this for charged metallic objects. We will derive a boundary condition on **E**, showing that the discontinuity in **E** is the surface charge density.

We will get these results much more easily with Gauss's Law later.

Can use superposition in another interesting way. Suppose we have an annulus between r = a and r = b which carries uniform charge density ρ_0. This charge distribution is physically equivalent to two spheres of equal and opposite charge density $\pm\rho_0$, one of radius a and the other of radius b. The potential and electric fields everywhere are simply the sum of the solutions for the two spherical charge distributions individually:



r > b $\qquad \varphi(\mathbf{r}) = \rho_0 \frac{4\pi}{3} b^3 \frac{1}{r} - \rho_0 \frac{4\pi}{3} a^3 \frac{1}{r} = \frac{Q}{r}$

$Q = \text{net charge} = \rho_0 \frac{4\pi}{3}\left(b^3 - a^3\right)$

$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}) = \frac{Q}{r^2}\,\hat{\mathbf{r}}$

a < r < b $\qquad \varphi(\mathbf{r}) = \rho_0\left(2\pi b^2 - \frac{2}{3}\pi r^2\right) - \rho_0 \frac{4\pi}{3} a^3 \frac{1}{r} = 2\pi\rho_0\left(b^2 - \frac{r^2}{3} - \frac{2}{3}\frac{a^3}{r}\right)$

$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}) = \rho_0 \frac{4\pi}{3}\mathbf{r} - \rho_0 \frac{4\pi}{3} a^3 \frac{1}{r^2}\,\hat{\mathbf{r}} = \rho_0 \frac{4\pi}{3}\left(1 - \frac{a^3}{r^3}\right)\mathbf{r}$

r < a $\qquad \varphi(\mathbf{r}) = \rho_0\left(2\pi b^2 - \frac{2}{3}\pi r^2\right) - \rho_0\left(2\pi a^2 - \frac{2}{3}\pi r^2\right) = 2\pi\rho_0\left(b^2 - a^2\right)$

$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}) = 0$

Why bother with $\varphi(\mathbf{r})$?

- $\mathbf{E}$ has vectors which make geometry difficult

• denominators in φ are $|\mathbf{r} - \mathbf{r}'|$, much easier to integrate than $|\mathbf{r} - \mathbf{r}'|^2$ or $|\mathbf{r} - \mathbf{r}'|^3$

Solving problems is easier.

Gauss's law

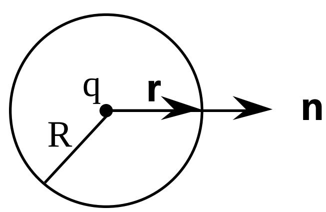
$$\int_S \mathbf{E} \cdot \mathbf{n} \, da = 4\pi \left(\text{charge enclosed}\right) = 4\pi \int_V \rho(\mathbf{r}) \, d^3r$$

• mathematical statement, not physics statement

• works because of inverse square law for **E**, and central nature of force

• also true for gravitational field

simple case: point charge at origin



$$\int_S \mathbf{E} \cdot \mathbf{n} \, da = \int_S \frac{q}{r^3} \mathbf{r} \cdot \mathbf{n} \, da = \frac{q}{R^2} \int_0^{2\pi} \int_0^{\pi} R^2 \sin\theta \, d\theta \, d\varphi = \frac{q}{R^2} 2\pi R^2 \int_0^{\pi} \sin\theta \, d\theta$$

$$2\pi q \left(- \cos\theta\right)\Big|_{\pi}^{0} = 2\pi q \left(- \left(- 1 - 1\right)\right) = 4\pi q$$

arbitrary surface: $\displaystyle\int_S \mathbf{E} \cdot \mathbf{n} \, da = \int_S \frac{q}{r^2} \hat{\mathbf{r}} \cdot \mathbf{n} \, da$

$\hat{\mathbf{r}} \cdot \mathbf{n} \, da$ = projection of da on locally spherical surface element

$= r^2 \, d\Omega$ full argument in Jackson

$$\int_S \mathbf{E} \cdot \mathbf{n} \, da = \int_S \frac{q}{r^2} r^2 \, d\Omega = q \int d\Omega = 4\pi q$$

why it works:

- $\mathbf{E}$ is central force, so that $\hat{\mathbf{r}} \cdot \mathbf{n}$ da projects da onto locally spherical surface element, i. e., $\hat{\mathbf{r}}$

- $\mathbf{E}$ is inverse square → r^2 factors from da and from $\mathbf{E}$ cancel

- both $\mathbf{n}$ and $\hat{\mathbf{r}}$ change sign on opposite sides of surface → integrand is positive everywhere

amazing mathematical fact: if q is outside surface, $\int_S \mathbf{E} \cdot \mathbf{n} \, da = 0$

Why? $\mathbf{E}$ does not change sign on opposite sides of surface → $\hat{\mathbf{r}} \cdot \mathbf{n}$ changes sign on opposite sides of surface → integral is 0

Examples of use of Gauss Law for solving problems:

$$\int_S \mathbf{E} \cdot \mathbf{n} \, da = 4\pi \left(\text{charge enclosed}\right)$$

Uniformly charged sphere:

Symmetry: $\mathbf{E}$ cannot depend on angles, only on r

$\mathbf{E}$ has only a radial component. All others are cancelled by symmetrically placed charge elements.

r < R $\quad E_r \, 4\pi r^2 = 4\pi \rho_0 \frac{4\pi}{3} r^3$

$$E_r = \frac{4\pi}{3} \rho_0 \, r$$

$$\mathbf{E} = \frac{4\pi}{3} \rho_0 \, \mathbf{r}$$

r > R $\quad E_r \, 4\pi r^2 = 4\pi \rho_0 \frac{4\pi}{3} R^3$

$$E_r = \frac{\rho_0 \frac{4\pi}{3} R^3}{r^2}$$

$$\mathbf{E} = \frac{\rho_0 \frac{4\pi}{3} R^3}{r^3}\,\mathbf{r}$$

Looks like a point charge for r > R.

What about potential?

$\mathbf{E} = -\nabla\varphi$

r < R : $\quad E_r = \frac{4\pi}{3}\rho_0\, r = -\frac{\partial\varphi}{\partial r} \quad\rightarrow\quad \varphi = -\frac{4\pi}{3}\rho_0 \frac{r^2}{2} + C_1$

r > R : $\quad E_r = \frac{\rho_0 \frac{4\pi}{3} R^3}{r^2} = -\frac{\partial\varphi}{\partial r} \rightarrow \quad \varphi = \frac{\rho_0 \frac{4\pi}{3} R^3}{r} + C_2$

Must have φ continuous everywhere, otherwise $\mathbf{E} = -\nabla\varphi$ will be infinite. Not physically possible. Adjust arbitrary constants of integration to achieve continuity.

Even after achieving continuity, there is one arbitrary constant still adjustable: overall constant will not affect the derivative, which is the physically meaningful quantity. So, one constant can be adjusted for convenience. Ususally, the convention is $\varphi \rightarrow 0$ as $r \rightarrow \infty$.

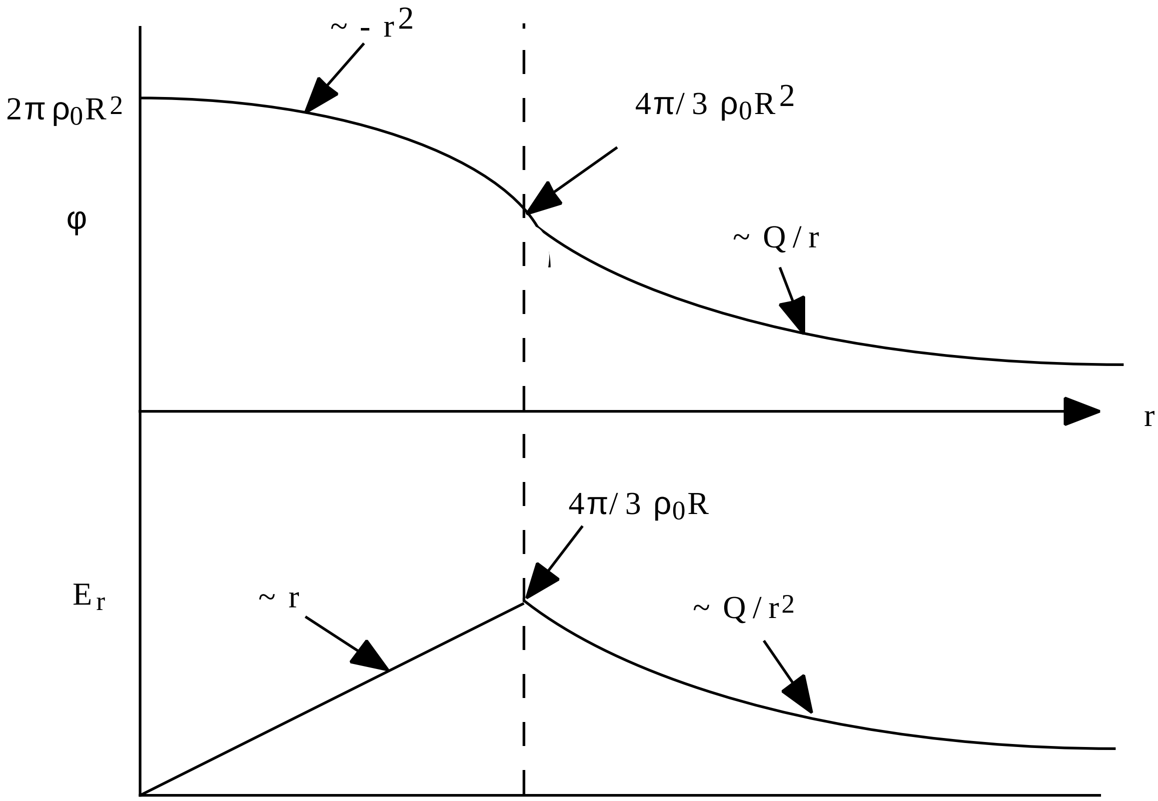
For r > R, this is achieved by $C_2 = 0$. Then φ looks like usual potential due to point charge at origin.

At r = R, adjust C_1 for continuity:

$$-\frac{4\pi}{3}\rho_0\frac{R^2}{2} + C_1 = \frac{\rho_0 \frac{4\pi}{3} R^3}{R} = \rho_0 \frac{4\pi}{3} R^2$$

$$C_1 = \frac{3}{2}\frac{4\pi}{3}\rho_0 R^2 = 2\pi\rho_0 R^2$$

r < R : $\quad \varphi(\mathbf{r}) = -\frac{4\pi}{3}\rho_0\frac{r^2}{2} + 2\pi\rho_0 R^2 = 2\pi\rho_0\left(R^2 - \frac{r^2}{3}\right)$



Line charge: $$\int_S \mathbf{E}\cdot\mathbf{n}\, da = 4\pi\left(\text{charge enclosed}\right)$$

$$E_r\, 2\pi r L = 4\pi\lambda L$$

$$E_r = \frac{2\lambda}{r}$$

$$\mathbf{E} = \frac{2\lambda}{r^2}\,\mathbf{r}$$

Same result as earlier with integration, but much more direct.

$$-\frac{\partial\varphi}{\partial r} = E_r = \frac{2\lambda}{r}$$

$$\varphi(\mathbf{r}) = -\,2\lambda \ln r + \text{const}$$

Sheet of charge: $$\int_S \mathbf{E}\cdot\mathbf{n}\, da = 4\pi\left(\text{charge enclosed}\right)$$

$$E_n A + E_n A = 4\pi\sigma A$$

$$E_n = 2\pi\sigma$$

$$\mathbf{E} = 2\pi\sigma\,\mathbf{n}$$

Microscopic form of Coulomb's law:

Gauss Law: $$\int_S \mathbf{E}\cdot\mathbf{n}\,da = 4\pi\left(\text{charge enclosed}\right) = 4\pi\int_V \rho(\mathbf{r})\,d^3r$$

Immediately use Divergence theorem:

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \int_V \nabla\cdot\mathbf{E}\,d^3r$$

$$\int_V \left(\nabla\cdot\mathbf{E} - 4\pi\rho\right) d^3r = 0$$

Holds for arbitrary volume   integrand = 0

If not, then draw a small surface around region where integrand ≠ 0, and the integral would be finite over this volume.

$$\nabla\cdot\mathbf{E} = 4\pi\rho$$ differential form of Gauss's law/Coulomb's law

Two Maxwell equations for **E**:

$$\nabla\cdot\mathbf{E} = 4\pi\rho$$

$$\nabla\times\mathbf{E} = 0$$

- fluid interpretation: if **E** were a velocity field for a fluid, ρ is the source of **E**

  lines of **E** begin on positive charge, end on negative charge

- holds for electrostatics, $\mathbf{E}(\mathbf{r})$. If $\mathbf{E}(\mathbf{r},t)$, then $\nabla\times\mathbf{E}$ has to be modified.

Potential form:

$$\nabla\cdot\mathbf{E} = \nabla\cdot\left(-\nabla\varphi\right) = -\ {}^2\varphi = 4\pi\rho \qquad {}^2\varphi = -4\pi\rho$$ Poisson's eq

if $\rho = 0$ $\quad {}^2\varphi = 0$ Laplace's eq

second Maxwell equation automatically satisfied:

$\nabla \times (\nabla\varphi) = 0$

Conductors and insulators

Conductor: exerts no force on a charge charges move freely until they come to equilibium, where $\mathbf{F} = 0$.

$\mathbf{F} = 0 \quad \mathbf{E} = \dfrac{\mathbf{F}}{q} = 0$ no electric field inside conductor, $\mathbf{E} = 0$.

Where are the charges in a conductor?

Physical answer: charges repel by Coulomb's law they go as far from each other as possible. They separate until they hit the surface of the conductor, where they remain.

Mathematical answer: $\mathbf{E} = 0 \quad \rho = \dfrac{\nabla \cdot \mathbf{E}}{4\pi} = 0 \quad \rho = 0$ inside conductor

Surface can exert force on charge by trapping it, so $\mathbf{E} \neq 0$ at surface. However, surface cannot exert a tangential force, so $F_t = 0 \quad E_t = 0$, $E_n \neq 0$.

Insulator:

charge fixed material exerts force on charge to overcome Coulomb repulsion.

Restoring force not infinite, charge can displace slightly from its rest position if it feels a field $\mathbf{E}$.

Displacement described by $\mathbf{P}$ = electric dipole polarization.

Dielectrics = insulators in which charge is allowed to displace slightly.

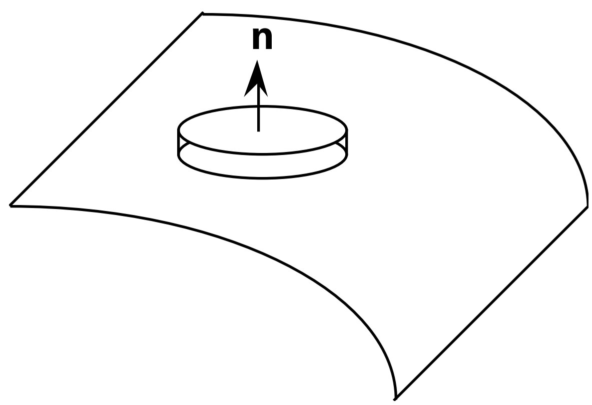
For the moment we will not be concerned with insulators or dielectrics, except in that they can hold a fixed volume charge distribution $\rho(\mathbf{r})$.

Boundary condition on conductor:

Physical reasoning charge resides on surface of conductor, magnitude of σ unknown

Mathematics using Maxwell's eq quantitative expression for σ

Normal component of **E**: draw a Gaussian surface



$$\int_S \mathbf{E} \cdot \mathbf{n}\, da = 4\pi \int_V \rho(\mathbf{r})\, d^3r$$

Shrink area of box so that **E** is essesntailly constant over surface

Shrink height so that only charge on surface is included in volume integral

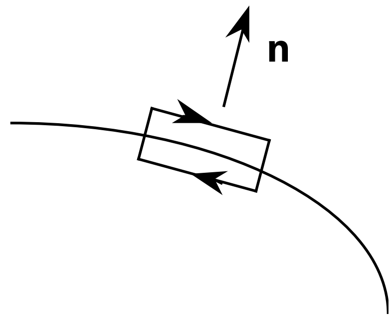
E = 0 inside conductor

$$E_n A = 4\pi \sigma A$$

$$E_n = -\frac{\partial \varphi}{\partial n} = 4\pi \sigma \qquad \frac{\partial \varphi}{\partial n} = \nabla\varphi \cdot \mathbf{n} = \text{normal gradient}$$

n = outward drawn normal

Tangential component: draw Stokes contour



Shrink segments || **n** until only surface is enclosed

Shrink length until **E** is essentially constant

$$0 = \int_S \nabla \times \mathbf{E} \cdot \mathbf{n} \, da = \int_C \mathbf{E} \cdot d\mathbf{l} = E_t L \qquad E_t = \text{outside conductor}$$

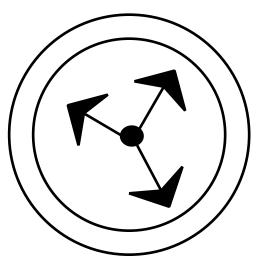
↑

Stokes' theorem: S is surface bounded by C

$E_t = 0$ outside the surface of conductor

This introduces the idea of *induced charge*. If there is a conductor in an electric field, there will be an *induced* surface charge on the conductor, whose magnitude is given by the normal component of the field.

Example: point charge at center of hollow sphere.



Gaussian surfaces inside, in annulus, and outside sphere give **E** in all cases.

$\sigma = \frac{1}{4\pi} E_n$ gives induced charge.

Formal properties of φ and **E**

All the physics of electrostatics given by above. Considerable mathematical development, illuminating nature of solutions, done since 1812, when Poisson introduced the potential, borrowed from gravitation. We will examine some of these general properties, and make a central formal solution applicable to all problems.

Dirac δ function: simplifies all the formal manipulations. Introduced by Dirac in 1920's to deal with formal development of quantum mechanics, equally useful for classical theory.

Imagine a function $\delta(\mathbf{r})$ (not really a function, since it is not differentiable everywhere, but a *distribution*) such that:

$$\delta(\mathbf{r}) = 0 \quad \text{everywhere except at } \mathbf{r} = 0$$

$$\int_V \delta(\mathbf{r})\, d^3r = 1$$

$$\int_V f(\mathbf{r})\, \delta(\mathbf{r} - \mathbf{r}_0)\, d^3r = f(\mathbf{r}_0)$$

$\delta(\mathbf{r})$ not differentiable at $\mathbf{r} = 0$

Example: point charge q at origin $\rho(\mathbf{r}) = q\, \delta(\mathbf{r})$

charge density infinite at point charge

total charge = $\int_V \rho(\mathbf{r})\, d^3r = q \int_V \delta(\mathbf{r})\, d^3r = q$

Example 1: $$\lim_{\sigma \to 0} \frac{e^{-\frac{x^2}{\sigma^2}}}{\sigma\sqrt{\pi}}$$

integral table: $$\int_{-\infty}^{\infty} x^{2a}\, e^{-\frac{x^2}{\sigma^2}}\, dx = \frac{1\cdot 3\cdot 5\cdot\cdot\cdot(2a-1)}{2^a}\, \sigma^{2a+1}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}}\, dx = \sigma\sqrt{\pi}$$

Normalization:

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{\sigma^2}}}{\sigma\sqrt{\pi}}\, dx = \frac{\sigma\sqrt{\pi}}{\sigma\sqrt{\pi}} = 1$$

Value at $x \neq 0$: $\lim\limits_{\sigma\to 0} \dfrac{e^{-\frac{x^2}{\sigma^2}}}{\sigma} = \dfrac{0}{0} = 0$ Note exponential goes to zero faster than any power. Alternate proof: Expand exponential

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\frac{e^{-\frac{x^2}{\sigma^2}}}{\sqrt{\pi}\,\sigma} = \frac{1}{\sqrt{\pi}\,\sigma\, e^{\frac{x^2}{\sigma^2}}} = \frac{1}{\sqrt{\pi}\,\sigma\left\{1 + \frac{x^2}{\sigma^2} + \frac{1}{2}\left(\frac{x^2}{\sigma^2}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{\sigma^2}\right)^3 + \frac{1}{n!}\left(\frac{x^2}{\sigma^2}\right)^n + \ldots\right\}}$$

$$= \frac{1}{\sqrt{\pi}\left\{\sigma + \frac{x^2}{\sigma} + \frac{1}{2}\frac{x^4}{\sigma^3} + \frac{1}{3!}\frac{x^6}{\sigma^5} + \ldots + \frac{1}{n!}\frac{x^n}{\sigma^{n-1}} + \ldots\right\}}$$

$$\begin{array}{lccccc} & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \sigma\to 0 & 0 & \infty & \infty & \infty & \infty \end{array}$$

$\to 0$

Integrals of arbitrary f(x):

$$\int_{-\infty}^{\infty} f(x)\, \frac{e^{-\frac{x^2}{\sigma^2}}}{\sigma\sqrt{\pi}}\, dx \;=\; \int_{-\infty}^{\infty} \left\{ f(0) + \left.\frac{df}{dx}\right|_0 x + \frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_0 x^2 + \ldots \right\} \frac{e^{-\frac{x^2}{\sigma^2}}}{\sigma\sqrt{\pi}}\, dx$$

$$\uparrow \qquad\qquad \uparrow$$

$$\text{odd integrand=0} \qquad \frac{\left(\frac{\sigma^3\sqrt{\pi}}{2}\right)}{\sigma\sqrt{\pi}}$$

All odd terms = 0 by symmetry,

all even terms of form $\dfrac{\sigma^{2a-1}}{\sigma} \to 0$ as $\sigma \to 0$

Therefore, $$\lim_{\sigma\to 0} \int_{-\infty}^{\infty} f(x)\, \frac{e^{-\frac{x^2}{\sigma^2}}}{\sigma\sqrt{\pi}}\, dx = f(0)$$

Example 2: $$\delta(\mathbf{r}) = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right)$$

$$\delta(\mathbf{r} - \mathbf{r}_0) = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right)$$

Consider a point charge in Poisson's equation:

$$\nabla^2\varphi(\mathbf{r}) = -4\pi\,\rho(\mathbf{r})$$

$$\nabla^2\left(\frac{q}{r}\right) = -4\pi\, q\,\delta(\mathbf{r})$$

Note vector identities which allow calculation of $\nabla^2\left(\frac{1}{r}\right)$ at $\mathbf{r} \neq 0$:

$$\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\left(\nabla\left(\frac{1}{r}\right)\right) = -\nabla\cdot\left(\frac{1}{r^3}\,\mathbf{r}\right)$$

$$= -\left\{\frac{1}{r^3}\nabla\cdot\mathbf{r} + \mathbf{r}\cdot\nabla\left(\frac{1}{r^3}\right)\right\} \qquad \nabla\cdot\left(f\mathbf{A}\right) = f\,\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$$

$$\uparrow \qquad\qquad \uparrow$$

$3 \qquad -\frac{3}{r^4}\hat{\mathbf{r}}$

$$= -\left\{ \frac{3}{r^3} - \frac{3}{r^3} \right\} = 0$$ except at $\mathbf{r} = 0$ where the derivatives are not defined

Normalization:

$$\int \nabla^2 \left(\frac{q}{r}\right) d^3r = \int \nabla \cdot \nabla \left(\frac{q}{r}\right) d^3r = -\int \nabla \cdot \mathbf{E}\, d^3r = -\int \mathbf{n} \cdot \mathbf{E}\, ds = -4\pi q$$

↑ divergence thm ↑ Gauss law

$$\rightarrow \quad \int -\frac{1}{4\pi} \nabla^2 \left(\frac{1}{r}\right) d^3r = 1$$

Formal relations in potential theory

Green's theorem

Mean value theorem

Uniqueness theorem

Formal solution to all electrostatics problems (Green's functions)

Green's theorem

Let $\mathbf{A} = \psi \nabla \varphi$ $\qquad$ ψ, φ are arbitrary functions of space

$$\nabla \cdot \mathbf{A} = \psi \nabla \cdot \nabla \varphi + \nabla \varphi \cdot \nabla \psi$$ product rule

$$= \psi \nabla^2 \varphi + \nabla \varphi \cdot \nabla \psi$$

Divergence theorem: $$\int_V \nabla \cdot \mathbf{A}\, d^3r = \int_S \mathbf{n} \cdot \mathbf{A}\, dS = \int_S \psi\, \mathbf{n} \cdot \nabla \varphi\, dS$$

$$= \oint_S \psi \frac{\partial \varphi}{\partial n}\, dS$$

normal derivative: $\mathbf{n} \cdot \nabla \varphi = \dfrac{\partial \varphi}{\partial n}$

Green's first identity:

$$\int_V \psi \;{}^2 \varphi + \nabla \varphi \cdot \nabla \psi \; d^3r = \oint_S \psi \frac{\partial \varphi}{\partial n}\, dS$$

Green's second identity: interchange ψ, φ and subtract

$$\int_V \psi \;{}^2 \varphi - \varphi \;{}^2 \psi \;\; d^3r = \oint_S \psi \frac{\partial \varphi}{d\, n} - \varphi \frac{\partial \psi}{\partial n} \;\; dS$$

Use this identity many times to derive formal relationships

Example: "prove" Gauss's law from Poisson's equation. (Of course, Poisson's equation is equivalent to Gauss's law, as we showed in class.)

Can use either Green's first or second identity.

Let ψ = const, ${}^2\varphi(\mathbf{r}) = -4\pi\rho(\mathbf{r})$. Then $\nabla\psi = 0 = {}^2\psi$, so 2nd term in LHS of either is zero, and 2nd term in Green's second identity is zero.

Substitute ${}^2\varphi(\mathbf{r}) = -4\pi\rho(\mathbf{r})$, and divide by y which appears on both sides, and

$$\int_V -4\pi\rho(\mathbf{r})\; d^3r = \oint_S \frac{\partial \varphi}{\partial n}\, dS = \int_S \nabla \varphi \cdot \mathbf{n}\; dS = \int_S -\mathbf{E} \cdot \mathbf{n}\; dS \qquad \text{or,}$$

$$\int_S \mathbf{E} \cdot \mathbf{n}\; da = 4\pi \int_V \rho(\mathbf{r})\; d^3r$$

Mean value theorem:

If $\nabla^2 \varphi = 0$ then $\varphi(\mathbf{r}_0) = \oint_S \frac{1}{4\pi R^2}\varphi(S)\, dS$

where S is a sphere of radius R about $\mathbf{r}_0$

Use Green's theorem twice.

Step 1: Let $\nabla^2 \varphi = 0$, let $\psi = \text{const.}$

Use Green's theorem $\int_V \left(\psi \nabla^2 \varphi - \varphi \nabla^2 \psi\right) d^3r = \oint_S \left(\psi \frac{\partial \varphi}{d\, n} - \varphi \frac{\partial \psi}{\partial\, n}\right) dS$

$\psi = \text{const}$ $\quad\quad \nabla^2 \psi = 0 \quad\quad \text{LHS} = 0$

$$\frac{\partial \Psi}{\partial\, n} = 0 \qquad 0 = \oint_S \psi \frac{\partial \varphi}{d\, n}\, dS = \oint_S \frac{\partial \varphi}{d\, n}\, dS$$

$$= \int_S -\mathbf{E} \cdot \mathbf{n}\, dS$$

"if φ is a solution to Laplace's equation, its normal derivative integrates to zero on any closed surface."

Consequence of Gauss's law, where there are no enclosed charges because the potential satisfies Laplaces's equation.

Step 2: let $\nabla^2 \varphi = 0$, $\psi = \frac{1}{|\mathbf{r} - \mathbf{r}_0|}$

Use Green's theorem $\int_V \left(\psi \nabla^2 \varphi - \varphi \nabla^2 \psi\right) d^3r = \oint_S \left(\psi \frac{\partial \varphi}{d\, n} - \varphi \frac{\partial \psi}{\partial\, n}\right) dS$

$$\int_V \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \nabla^2 \varphi - \varphi \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right)\right) d^3r = \oint_S \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \frac{\partial \varphi}{\partial\, n} - \varphi \frac{\partial}{\partial\, n}\left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right)\right) dS$$

Let $\mathbf{r}$ be on a sphere surrounding $\mathbf{r}_0$ $\quad |\mathbf{r} - \mathbf{r}_0| = R$

$$\mathbf{r} - \mathbf{r}_0 = \mathbf{r}' \quad \frac{\partial}{\partial n} = \frac{\partial}{\partial r'} \qquad \frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right) = \frac{\partial}{\partial r'}\left(\frac{1}{r'}\right) = -\frac{1}{r'^2} = -\frac{1}{R^2} \text{ for } \mathbf{r} \text{ on } S$$

$$\text{RHS} = \oint_S \left(\frac{1}{R}\frac{\partial \varphi}{\partial n} - \varphi\left(\frac{1}{R^2}\right)\right) dS = 0 + \frac{1}{R^2}\int_S \varphi \, dS$$

$\uparrow$

0 by previous result

$$\text{LHS} = \int_V \frac{1}{|\mathbf{r} - \mathbf{r}_0|}\;{}^2\varphi - \varphi\;{}^2\left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right) d^3r$$

$$= \int_V \left\{ \quad 0 \quad - \varphi\big(-4\pi\,\delta(\mathbf{r} - \mathbf{r}_0)\big)\right\} d^3r = 4\pi\,\varphi(\mathbf{r}_0)$$

Equating the two sides,

$$4\pi\,\varphi(\mathbf{r}_0) = \frac{1}{R^2}\int_S \varphi \, dS \qquad \text{or} \qquad \varphi(\mathbf{r}_0) = \frac{1}{4\pi R^2}\int_S \varphi(S)\, dS$$

Use this to prove

1. Solutions to Laplace's equation have no maxima or minima at interior points. (If there were interior extrema, draw a sphere around them, and they would violate the above argument.)

2. For any closed surface, both maximum and minimum values must be on the surface. (No interior extrema.)

3. If $\varphi = 0$ on a surface, then $\varphi = 0$ inside the surface as well. (Maximum and minimum = 0, only solution is $\varphi = 0$.)

Uniqueness theorem:

Solution to Poisson's equation is uniquely specified by either

1) value of φ on boundary (Dirichlet boundary conditions) or

2) value of $\frac{\partial \varphi}{\partial n}$ on boundary (Neumann boundary conditions)

proof: let φ_1 and φ_2 be two solutions to Poisson's equation

$$^2 \varphi_1(\mathbf{r}) = - 4\pi \rho(\mathbf{r})$$

$$^2 \varphi_2(\mathbf{r}) = - 4\pi \rho(\mathbf{r})$$ for the same charge density $\rho(\mathbf{r})$

Use Green's first identity, with $\psi = \varphi = \varphi_1 - \varphi_2$

$$\int_V \psi \;\; ^2 \varphi + \nabla \varphi \cdot \nabla \psi \, d^3r = \oint_S \psi \frac{\partial \varphi}{\partial n} dS$$ Green's first identity

$$\int_V \left(\varphi_1 - \varphi_2\right) \;\; ^2 \left(\varphi_1 - \varphi_2\right) + \nabla \left(\varphi_1 - \varphi_2\right) \cdot \nabla \left(\varphi_1 - \varphi_2\right) d^3r = \int_S \left(\varphi_1 - \varphi_2\right) \frac{\partial}{\partial n}\left(\varphi_1 - \varphi_2\right) dS$$

$\uparrow$ $\uparrow$

$- 4\pi \rho(\mathbf{r}) + 4\pi \rho(\mathbf{r}) = 0$ $\left|\nabla \left(\varphi_1 - \varphi_2\right)\right|^2$

$$\int_V \left|\nabla \left(\varphi_1 - \varphi_2\right)\right|^2 d^3r = \int_S \left(\varphi_1 - \varphi_2\right) \frac{\partial}{\partial n}\left(\varphi_1 - \varphi_2\right) dS$$

If either 1) or 2) is satisfied, RHS = 0, $\nabla \left(\varphi_1 - \varphi_2\right) = 0$

$\varphi_1 - \varphi_2 =$ constant

Significance:

φ and $\mathbf{E}$ are *uniquely* determined by

1) charge distribution $\rho(\mathbf{r})$

2) one of two boundary conditions

φ on surface (Dirichlet)

$\mathbf{E} \cdot \mathbf{n} = - \frac{\partial \varphi}{\partial n}$ on surface (Neumann)

Example: Let $\varphi = 0$ on surface, and let $\nabla^2 \varphi = 0$ throughout volume. Then $\varphi = 0$ is the only solution throughout the volume.

proof: $\varphi = 0$ is a solution satisfying the boundary conditions, therefore it is the only solution.

Very useful result: no matter how you find the solution, by guessing or otherwise, you can stop working. The solution is unique, there is nothing more to find.

Formal solution to electrostatics problems

Often not very helpful in practice, because the solutions are in terms of integrals that can be extremely difficult, if not impossible. But, it gives a useful way of looking at the solutions, and the technique is used extensively elsewhere in physics.

Two steps, each using Green's theorem: (1) use the "propagation function" $\psi = \dfrac{1}{|\mathbf{r} - \mathbf{r}'|}$, which gives a beautiful physical interpretation of the solution $\varphi(\mathbf{r})$ in terms of the charge distribution $\rho(\mathbf{r})$. However, this will not be the most useful solution, because it doesn't treat surfaces with induced charges cleverly. (2) use a modified propagation function $\dfrac{1}{|\mathbf{r} - \mathbf{r}'|} + \chi(\mathbf{r}, \mathbf{r}')$ where χ is a solution to Laplace's equation. χ is adjusted to make the solution in the presence of surfaces easy.

Motivation:

We know one solution, based on superposition and the potential of a point charge.

$$\varphi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

Interpretation: $\frac{1}{|\mathbf{r} - \mathbf{r}'|}$ gives the potential at $\mathbf{r}$ due to a unit point charge at $\mathbf{r}'$. The total potential at $\mathbf{r}$ is then the sum of the contributions from all sources at $\mathbf{r}'$, weighted by the strength of the source at $\mathbf{r}'$. $\frac{1}{|\mathbf{r} - \mathbf{r}'|}$ is sometimes called the propagation function or the Green's function.

This works perfectly if there are no induced charges on conducting surfaces, which will also produce a contribution to the potential at $\mathbf{r}$.

Step 1: To get a formal solution when surfaces are important, use Green's theorem.

Let $\psi = \frac{1}{|\mathbf{r} - \mathbf{r}'|}$ $\qquad \nabla^2 \varphi = -4\pi\,\rho(\mathbf{r})$

Green's theorem: $\int_V \left(\psi\,\nabla^2\varphi - \varphi\,\nabla^2\psi\right) d^3r \;=\; \oint_S \left(\psi \frac{\partial\varphi}{d\,n} - \varphi \frac{\partial\psi}{\partial\,n}\right) dS$

integrate over primed variables

$$\int_V \left[\frac{-4\pi\,\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \varphi(\mathbf{r}')\,\nabla^2\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] d^3r' \;=\; \oint_{S'} \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial\varphi(\mathbf{r}')}{\partial\,n} - \varphi(\mathbf{r}')\frac{\partial}{\partial\,n}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] dS'$$

↑

$-4\pi\,\delta(\mathbf{r} - \mathbf{r}')$

$$-4\pi \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' + 4\pi\,\varphi(\mathbf{r}) \;=\; \text{above}$$

$$\varphi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' + \frac{1}{4\pi} \oint_{S'} \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial\varphi(\mathbf{r}')}{\partial\,n} - \varphi(\mathbf{r}')\frac{\partial}{\partial\,n}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] dS'$$

- Formal solution for $\varphi(\mathbf{r})$ given the charge distribution $\rho(\mathbf{r})$, including the effects of charges induced on the surfaces.
- If no surfaces, the solution reduces to the familiar superposition form.
- Basic trick: $\nabla^2\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -4\pi\,\delta(\mathbf{r} - \mathbf{r}')$

• Not yet best solution: seems to require knowing both φ and $\frac{\partial\varphi}{\partial n}$ to find the surface integral. Uniqueness theorem says only one of these two is needed to define the solution.

Step 2.

Generalize the "propagation function" to zero out one of the terms in the surface integral. Note if we add a solution of Laplace's equation to $\frac{1}{|\mathbf{r} - \mathbf{r}'|}$, we preserve the basic δ function equation:

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \chi(\mathbf{r}, \mathbf{r}') \right) = -4\pi\,\delta(\mathbf{r} - \mathbf{r}') \qquad \text{where} \quad \nabla^2 \chi(\mathbf{r}, \mathbf{r}') = 0 \,.$$

$\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \chi(\mathbf{r}, \mathbf{r}')$ is called a Green's function.

There are many Green's functions, all related by adding a solution to Laplace's equation.

Apply Green's theorem again, this time with $\psi = G(\mathbf{r}, \mathbf{r}')$ = Green's function. All previous steps go through:

$$\varphi(\mathbf{r}) = \int_V \rho(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, d^3r' \quad + \quad \frac{1}{4\pi} \oint_{S'} \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial \varphi(\mathbf{r}')}{\partial n} - \varphi(\mathbf{r}') \frac{\partial}{\partial n} G(\mathbf{r}, \mathbf{r}') \right] dS'$$

Now we have flexibility to choose $G(\mathbf{r}, \mathbf{r}')$ to simplify the surface integrals. If we are concerned with Dirichlet boundary conditions, where φ is given on S, choose $\chi(\mathbf{r}, \mathbf{r}')$ such that $G(\mathbf{r}, \mathbf{r}') = 0$ if $\mathbf{r}'$ is on S′. Then the solution is given above, as

$$\varphi(\mathbf{r}) = \int_V \rho(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, d^3r' \quad - \quad \frac{1}{4\pi} \int_{S'} \varphi(\mathbf{r}') \frac{\partial}{\partial n} G(\mathbf{r}, \mathbf{r}')\, dS'$$

If we deal only with grounded surfaces where $\varphi = 0$, as we will in this course, then the surface integral is completely zero, and

$$\varphi(\mathbf{r}) = \int_V \rho(\mathbf{r}') \, G(\mathbf{r}, \mathbf{r}') \, d^3r'$$

• $G(\mathbf{r}, \mathbf{r}')$ depends on the surfaces - we chose G to vanish on the surface

• once G is found for a given surface, all problems are solved, no matter what the charge distribution. Simply do the integral.

• Green's functions maintain the propagator idea: $G(\mathbf{r}, \mathbf{r}')$ is the potential at $\mathbf{r}$ due to a unit charge at $\mathbf{r}'$, including the effect of surfaces.

Method of Images: illustration of simple ideas, uniqueness theorem and of Green's functions

Motivation: Given a charge distribution and a boundary, try to solve the problem intuitively by placing fictitious charges outside the region of interest. If this produces a potential that satisfies the boundary conditions inside the region of interest, it is the only solution, by the uniqueness theorem. Also, it is equivalent to finding the Green's function, since the potential due to the fictitious charges will satisfy Laplace's equation except at the position of the charge.

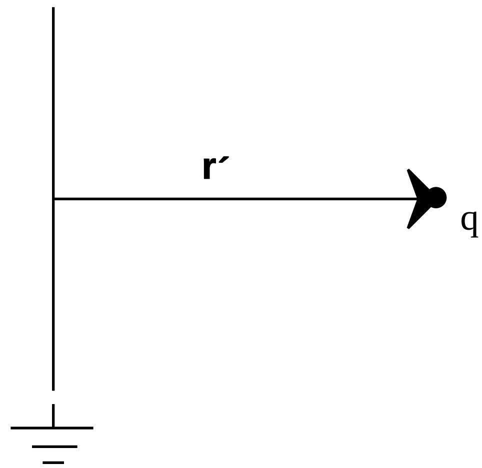
Example of an *equivalent* problem: different configurations of charge with same fields and potentials in region of interest. We will encounter several kinds of equivalent problems.

Two examples:

point charge near a grounded plane,

point charge near a grounded sphere.

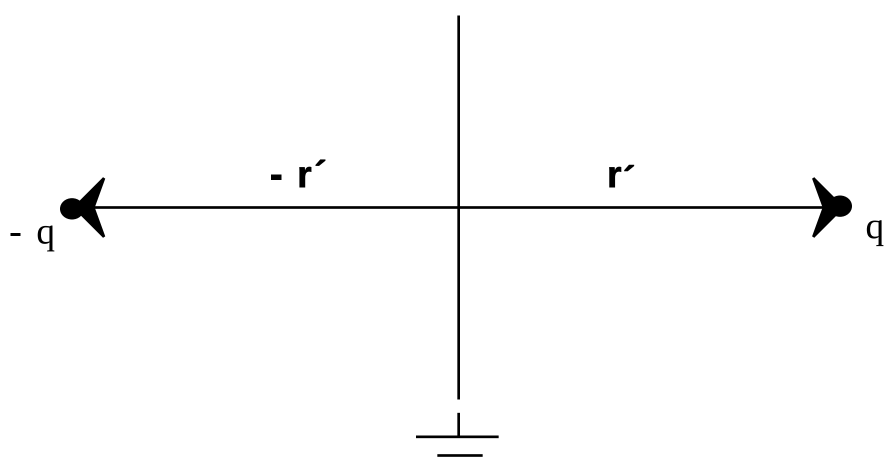
point charge near a grounded plane:



We want solution for $x > 0$ only. Satisfy boundary condition that $\varphi = 0$ on the plane.

Physical interpretation: **E** from point charge drives charge in plane to ground, leaves induced charge on plane. Induced charge creates its own **E** field. Total **E** is due to point charge + induced charge.

Boundary condition is easily satisfied by placing a fictitious charge symmetrically on other side of plane:



$$\varphi(\mathbf{r}) = \frac{q}{|\mathbf{r} - \mathbf{r}'|} - \frac{q}{|\mathbf{r} + \mathbf{r}'|} \qquad \text{(Choose origin so } \mathbf{r}' \parallel \mathbf{i}\text{)}$$

at $x = 0$, $\varphi(0, y, z) = \dfrac{q}{\sqrt{r'^2 + y^2 + z^2}} - \dfrac{q}{\sqrt{r'^2 + y^2 + z^2}} = 0.$

• $\varphi(\mathbf{r})$ satisfies the boundary condition, therefore it is the only solution to the problem.

• $\varphi(\mathbf{r})$ is not the solution in the half space $x < 0$. φ satisfies Laplace's equation for $x < 0$, and it is zero on the plane $x = 0$ and at ∞. Therefore, by the mean value theorem, $\varphi = 0$ everywhere in the half space $x < 0$.

• In the half space $x > 0$, the potential due to the fictitious charge $- q$ satisfies Laplace's equation. It is the term added to $\dfrac{1}{|\mathbf{r} - \mathbf{r}'|}$ to create the Green's function for the grounded plane.

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}''|} \quad \text{where } \mathbf{r}'' = - x'\,\mathbf{i} + y'\,\mathbf{j} + z'\,\mathbf{k}$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{\sqrt{(x + x')^2 + (y - y')^2 + (z - z')^2}}$$

Note that $G(\mathbf{r}, \mathbf{r}') = 0$ if $\mathbf{r}'$ is on the surface ($x' = 0$), as required for the Green's function.

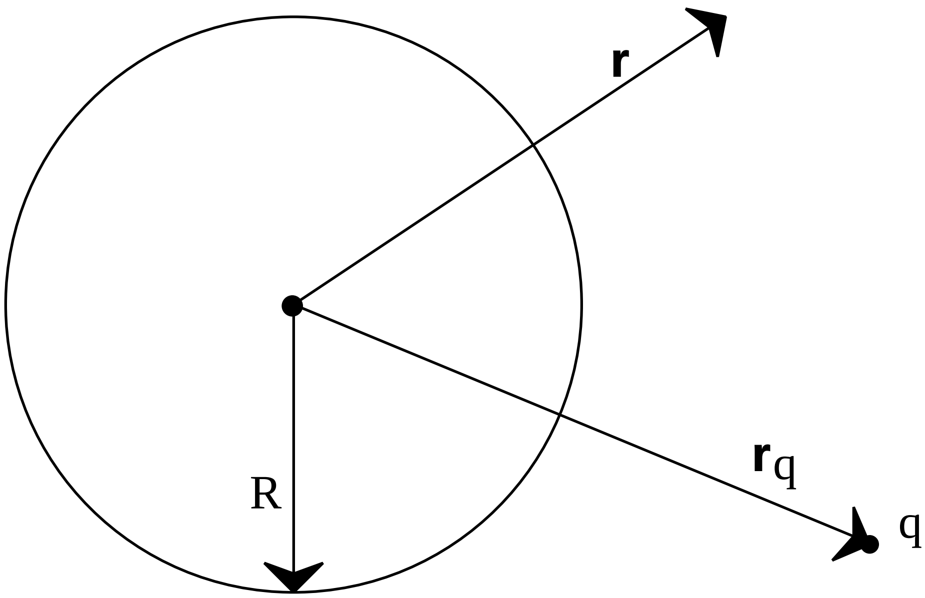
• The full solution, for an arbitrary charge distribution, is

$$\varphi(\mathbf{r}) = \int_V \rho(\mathbf{r}')\, G(\mathbf{r}, \mathbf{r}')\, d^3r$$

• The image of continuous charge distribution is the sum of the point images: a line charge has an image line charge, etc.

• By constructing the image solution for a point charge, we have the solution for an arbitrary charge distribution. This is the idea of the propagator: the solution is the superposition of the solutions for individual point charges making up the given charge distribution.

2) Point charge near a grounded conducting sphere



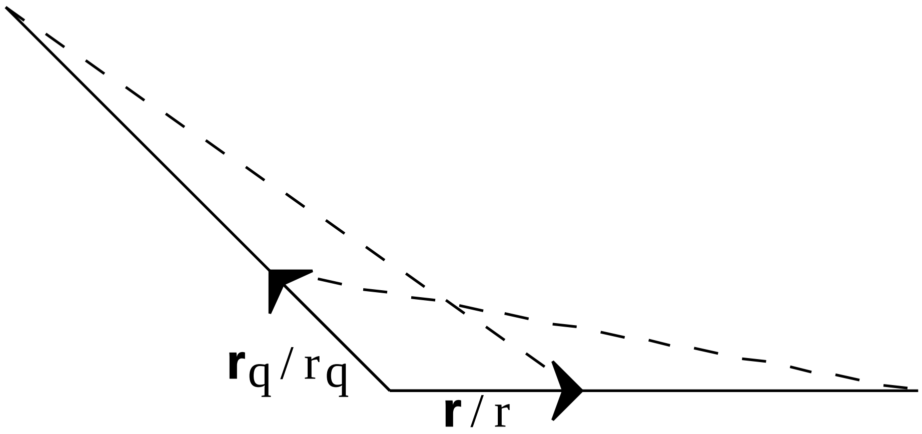
Boundary condition: $\varphi = 0$ on sphere, $|\mathbf{r}| = R$.

Expect image charge q' to be located on line from center to $\mathbf{r}_q$

$$\varphi(\mathbf{r}) = \frac{q}{|\mathbf{r} - \mathbf{r}_q|} + \frac{q'}{|\mathbf{r} - \alpha\, \mathbf{r}_q|}$$

For $|\mathbf{r}| = R$

$$\varphi(\mathbf{r}) = \frac{q}{R\left|\hat{\mathbf{r}} - \frac{r_q}{R}\,\hat{\mathbf{r}}_q\right|} + \frac{q'}{\alpha\, r_q\left|\frac{R}{\alpha\, r_q}\,\hat{\mathbf{r}} - \hat{\mathbf{r}}_q\right|}$$



Symmetry: $|b\,\hat{\mathbf{r}} - \hat{\mathbf{r}}_q| = |\hat{\mathbf{r}} - b\,\hat{\mathbf{r}}_q|$

Thus $\varphi(|\mathbf{r}| = R) = 0$ if

$$\frac{q}{R} = -\frac{q'}{\alpha\, r_q} \qquad \text{prefactors same}$$

$$\frac{r_q}{R} = \frac{R}{\alpha\, r_q} \qquad \text{vector lengths same}$$

Solution: $\alpha\, r_q = \dfrac{R^2}{r_q}$

$$q' = -\frac{R^2}{r_q}\,\frac{q}{R} = -q\,\frac{R}{r_q}$$

- As r_q recedes to ∞, q' goes to the origin and becomes 0
- As r_q approaches R, q' also approaches R and $q' \rightarrow -q$.
- If $r_q = R + \delta$, $\alpha\, r_q = \dfrac{R^2}{R+\delta} = R\left(\dfrac{1}{1+\dfrac{\delta}{R}}\right) = R\left(1 - \dfrac{\delta}{R}\right) = R - \delta$

ie, q' is symmetrically placed and the geometry is like the planar problem

Full solution:

$$\varphi(\mathbf{r}) = \frac{q}{\left|\mathbf{r} - \mathbf{r}_q\right|} - \frac{q\,\frac{R}{r_q}}{\left|\mathbf{r} - \frac{R^2}{r_q{}^2}\,\mathbf{r}_q\right|}$$

Separation of variables

Motivation: given $\varphi(\mathbf{r})$ on a surface, find $\varphi(\mathbf{r})$ everywhere. Works only for solutions to Laplace's equation, $\nabla^2\varphi = 0$.

Procedure: write solution as an expansion in some set of functions with unknown coefficients, then find coefficients by satisfying boundary conditions.

Along the way we will find useful sets of functions: sin, cos, sinh & cosh; Legendre functions and spherical harmonics, Bessel functions.
Expansions will make use of orthonormality and completeness. These concepts are central to all of physics.

Rectangular coordinates:

$$^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Assume $\varphi(x, y, z) = X(x)Y(y)Z(z)$. Not all functions are of this form,
e, g., $5x^2 + 6y$. It is *not* a product of functions of x and y. While this eliminates many functions, we will show through completeness that any function can be written in this way, by taking sums of terms that are of this form.

Partial differential equation simplifies to three ordinary differential equations:

$$\nabla^2\varphi = Y\,Z\,\frac{d^2X}{dx^2} + X\,Z\,\frac{d^2Y}{dy^2} + X\,Y\,\frac{d^2Z}{dz^2} = 0$$

Divide by XYZ:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

To satisfy equation, each term must be a constant, and all constants sum to zero.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2$$

$$\gamma^2 = \alpha^2 + \beta^2$$

Solutions are well known: $X = e^{\pm i\alpha x}$ $\quad Y = e^{\pm i\beta y}$ $\quad Z = e^{\pm\gamma z}$

General form: $\{A\sin\alpha x + B\cos\alpha x\}\,\{C\sin\beta y + D\cos\beta y\}\,\{E\sinh\gamma z + F\cosh\gamma z\}$

where α, β, and γ are arbitrary except for the condition $\gamma^2 = \alpha^2 + \beta^2$.

By taking sums of such terms, any function can be represented.

Example: let $\varphi(x, y) = f(x, y)$ a distance z_o above a grounded plane. Find γ everywhere.

Solution: $\varphi(z = 0) = 0 \Rightarrow F = 0$. Then

$$\varphi(x, y, z_o) = \{A\sin\alpha x + B\cos\alpha x\}\,\{C\sin\beta y + D\cos\beta y\}\ \sinh\gamma z_o$$

expand the function f(x,y) as Fourier series, and the problem is solved. If f(x,y) = sin5x + cos 8y, then

$$A = \frac{1}{\sinh\gamma z_o},\ B = 0,\ C = 0,\ D = \frac{1}{\sinh\gamma z_o},\ \alpha = 5,\ \beta = 8,\ \gamma = \sqrt{5^2 + 8^2}$$

$$\varphi = \left(\sin 5x + \cos 8y\right)\frac{\sinh\gamma z}{\sinh\gamma z_o}$$

See solution: x, y dependence is same at all z, its magnitude changes with z by $\sinh\gamma z$, going to zero at z = 0.

Note: by uniqueness theorem, this is the only solution.

Homework: take examples

General case: how to find coefficients? Use orthonormality.

Require that if $\{f_i(\mathbf{r})\}$ is the set of functions, then

$$\int_V d^3r\, f_i^*(\mathbf{r})\, f_j(\mathbf{r}) = \delta_{ij}$$

Then if we want to expand the function $F(\mathbf{r}) = \sum_i A_i f_i(\mathbf{r})$

$$\int_V d^3r\, f_j^*(\mathbf{r})\, F(\mathbf{r}) = \int_V d^3r\, f_j^*(\mathbf{r}) \sum_i A_i f_i(\mathbf{r})) =$$

$$\sum_i A_i \int_V d^3r\, f_j^*(\mathbf{r})\, f_i(\mathbf{r}) = \sum_i A_i \delta_{ij} = A_j$$

Example: sin and cos $\displaystyle\int_0^\pi dx \sin mx \sin nx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n \end{cases}$

How do we know any function can be expanded in the set?

Completeness means it can. Completeness can be expressed with the

δ function: if $\sum_i f_i^*(\mathbf{r})\, f_i(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$, then the set is complete.

Suppose that $F(\mathbf{r})$ is an arbitrary function. If it can be expanded, then

$$F(\mathbf{r}) = \sum_i A_i f_i(\mathbf{r}) = \sum_i \left(\int_V d^3r'\, f_i^*(\mathbf{r}')\, F(\mathbf{r}') \right) f_i(\mathbf{r})) \qquad \text{(orthogonality)}$$

$$= \int_V d^3r' \sum_i f_i^*(\mathbf{r}')\, f_i(\mathbf{r})\, F(\mathbf{r}') \qquad \text{(re-arrange integral \& sum)}$$

If $F(\mathbf{r})$ is an arbitrary function, then $\sum_i f_i^*(\mathbf{r}')\, f_i(\mathbf{r})$ must be $\delta(\mathbf{r} - \mathbf{r}')$. Conversely, if $\sum_i f_i^*(\mathbf{r}')\, f_i(\mathbf{r})$ is $\delta(\mathbf{r} - \mathbf{r}')$, then $F(\mathbf{r})$ can be an arbitrary function.

Completeness is hard to prove. We will use this fact but not prove it.

Spherical coordinates:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Assume $\Phi = \frac{U(r)}{r} P(\theta)\, Q(\varphi)$, substitute.

$$PQ\frac{d^2U}{dr^2} + \frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2} = 0$$

Multiply by $\frac{r^2\sin^2\theta}{UPQ}$:

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{r^2\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] + \frac{1}{Q}\frac{d^2Q}{d\varphi^2} = 0$$

↑ function of r,θ only ↑ function of φ only

Therefore,

$$\frac{1}{Q}\frac{d^2Q}{d\varphi^2} = -\,m^2$$

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{r^2\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right] = +\,m^2$$

Divide by $\sin^2\theta$:

$$r^2\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{\sin\theta}\frac{1}{P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0$$

↑ function of r only ↑ function of θ only

Therefore,

$$r^2 \frac{1}{U} \frac{d^2U}{dr^2} = l(l + 1)$$

$$\frac{1}{\sin\theta} \frac{1}{P} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = - l(l + 1)$$

where l(l + 1) is any real number. We will show that l must be positive integer later.

Rewriting in standard form by collecting terms:

$$\frac{d^2U}{dr^2} - \frac{l(l + 1)}{r^2} U = 0$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dP}{d\theta}\right) + \left\{ l(l + 1) - \frac{m^2}{\sin^2\theta} \right\} P = 0$$

Q equation is easily solved: $Q = e^{\pm im\varphi}$

m must be integer to be single valued.

U equation is easily solved: $U(r) = Ar^{l+1} + \frac{B}{r^l}$

Only non trivial equation is for θ.

In terms of $x = \cos\theta$, $\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} = - \sin\theta \frac{d}{dx}$

$$- \frac{d}{dx} \left(\sin^2\theta \frac{dP}{dx}\right) + \left\{ l(l + 1) - \frac{m^2}{\sin^2\theta} \right\} P = 0$$

$\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$

$$\frac{d}{dx} \left((1 - x^2) \frac{dP}{dx}\right) + \left\{ l(l + 1) - \frac{m^2}{1 - x^2} \right\} P = 0$$

Generalized Legendre equation

Solutions: Associated Legendre functions

Consider first $m = 0$ no dependence on φ. Azimuthal symmetry.

Ordinary Legendre equation.

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + l(l+1)P = 0$$

Perhaps a power series will work.

$$P(x) = x^{\alpha}\sum_{n=0}^{\infty} a_n x^n$$

Substitute $\Rightarrow$ $\dfrac{dP}{dx} = \alpha x^{\alpha-1}\displaystyle\sum_{n=0}^{\infty} a_n x^n + x^{\alpha}\sum_{n=0}^{\infty} a_n n x^{n-1}$

etc.

Write Legendre equation as a power series:

$$\sum_{n=0}^{\infty}(\qquad) x^n = 0$$

Then each coefficient of x^n must be zero.

$$\Rightarrow \quad a_{j+2} = \frac{(\alpha + j)(\alpha + j + 1) - l(l+1)}{(\alpha + j + 1)(\alpha + j + 2)} a_j$$ *recursion relation*

Start with a_o chosen, all a_n are determined by the recursion relation.

It turns out there are two choices:

$\alpha = 0 \Rightarrow$ even powers of x

$\alpha = 1 \Rightarrow$ odd powers of x

At $x = \pm 1$, the series diverges unless it terminates. This will happen only if l = integer. Then at $\alpha + j = l$, the coefficient of $a_{j+2} = 0$, and the series terminates.

By construction the first few solutions can be found:

$P_o(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

• The $P_l(x)$ are constructed to have $P_l(1) = 1$ by convention. They are not normalized.

Rodriguez formula: $P_l(x) = \frac{1}{2^l\, l!}\frac{d^l}{dx^l}(x^2 - 1)^l$

Used to prove lots of properties.

Example of manipulation using differential equation: prove orthogonality of P_l, $P_{l'}$, ie $\int_{-1}^{1} P_{l'}(x)\, P_l(x)\, dx = 0$.

by differential equation,

$$\int_{-1}^{1} P_{l'}(x)\left\{\frac{d}{dx}\left((1 - x^2)\frac{dP_l}{dx}\right) + l(l + 1)P_l(x)\right\} dx = 0$$

integrate by parts:

$$\int_{-1}^{1} \left(\frac{dP_{l'}}{dx}\right)(1 - x^2)\frac{dP_l}{dx}\, dx = l(l + 1)\int_{-1}^{1} P_{l'}(x)\, P_l(x)\, dx$$

notice LHS is symmetric in l, l'. Interchange l, l' and subtract:

$$0 = \{l(l + 1) - l'(l' + 1)\}\int_{-1}^{1} P_{l'}(x)\, P_l(x)\, dx$$

Thus if $l \neq l'$, the integral is zero. Orthogonality.

Integration of the Rodriguez formula $\Rightarrow$ $\int_{-1}^{1}\left(P_l(x)\right)^2 dx = \frac{2}{2l + 1}$

With this, we can find coefficients for expansion of an arbitrary function:

$$F(x) = \sum_{l=0}^{\infty} a_l\, P_l(x) \quad \Rightarrow \quad a_l = \frac{2l + 1}{2}\int_{-1}^{1} F(x)P_l(x)\, dx$$

Use P_l when there is azimuthal symmetry: φ is irrelevant and m = 0.

Then the general solution to Laplace's equation is

$$\Phi(\mathbf{r}) = \sum_{l=0}^{\infty} \frac{A_l r^{l+1} + B_l / r^l}{r} P_l(\cos\theta)$$

where $\dfrac{A_l r^{l+1} + B_l / r^l}{r}$ is $\dfrac{U(r)}{r}$

Example: $\Phi(\mathbf{r}) = \Phi(\theta)$ on surface of a sphere. Find $\Phi(\mathbf{r})$ everywhere.

Inside: Φ should be finite at the origin $\Rightarrow B_l = 0$ all l.

On sphere, $\Phi(a, \theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)$

$$A_l a^l = \frac{2l+1}{2} \int_0^{\pi} \Phi(a, \theta) P_l(\cos\theta) \sin\theta \, d\theta$$

This determines $\Phi(\mathbf{r})$ inside the sphere.

Outside, Φ must be finite at $r = \infty$. Therefore $A_l = 0$ for all l.

On sphere, $\Phi(a, \theta) = \sum_{l=0}^{\infty} B_l \frac{1}{a^{l+1}} P_l(\cos\theta)$

$$\frac{B_l}{a^{l+1}} = \frac{2l+1}{2} \int_0^{\pi} \Phi(a, \theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Homework: (i) find A_l, B_l for two cases: Φ is $\pm$ V on the top an bottom hemispheres, and Φ varies like $\cos^2\theta$ over the entire sphere.

(ii) Find the potential and field of a uncharged sphere placed in an initially uniform electric field. Realize the boundary conditions. Initially uniform means at ∞, $\mathbf{E}$ = const. Then on surface of sphere, $E_t = 0$. This is enough to find all the coefficients.

Homework:

Find the potential Φ and field $\mathbf{E}$ for an uncharged conducting sphere placed in an initially uniform electric field, using an expansion in Legendre polynomials. Choose the z axis to be the initially uniform field direction.

a. Write the most general solution to Laplace's equation in terms of the radial functions $\frac{U(r)}{r} = A_l r^l + B_l / r^{l+1}$ and the Legendre polynomials $P_l(\cos\theta)$. Since the problem has azimuthal symmetry, no spherical harmonics are required.

b. Use the boundary condition at $r = \infty$ to determine all the A_j.

c. Use the boundary condition at $r = a$ to determine all the B_j except B_0. What determines B_0?

d. With all coefficients determined, write explicit forms for φ and $\mathbf{E}$ for the space outside the sphere.

e. Determine the induced charge density on an initially uncharged sphere.

f. Determine the total induced charge on an initially uncharged sphere.

Solution

Choose the origin at the center of the sphere and the z axis along the initially uniform field direction, so that for $r = \infty$, $\mathbf{E} = E_0\mathbf{k}$. With this coordinate system, the problem has azimuthal symmetry, so a solution in terms of Legendre polynomials is possible.

a. Since there are no charges in the region outside the sphere, φ is a solution to Laplace's equation, $^2\varphi = 0$. Thus

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \left\{ A_l\, r^l + \frac{B_l}{r^{l+1}} \right\} P_l(\cos\theta)$$

b. The boundary condition at $r = \infty$ is $\mathbf{E} = -\nabla\varphi = E_0\mathbf{k}$. Along the z axis ($\theta = 0$),

$$\mathbf{E} = E_z\mathbf{k} \qquad E_z = -\left.\frac{\partial\varphi}{\partial r}\right|_{\theta=0} = -\sum_{l=0}^{\infty} \left\{ A_l\, l\, r^{l-1} - \left(l+1\right)\frac{B_l}{r^{l+2}} \right\} P_l(1)$$

Recall that $P_l(1) = 1$. the first few terms are

$$E_z = \qquad\qquad \frac{B_0}{r^2}$$

$$- A_1 + \frac{2B_1}{r^3}$$
$$- 2A_2 r + \frac{3B_2}{r^4}$$
$$+ \ldots$$

For large r, $E_z = E_0$ $\qquad - A_1 = E_0$

$$A_j = 0 \quad j > 1$$

The B_l are not determined by this BC.

c. On the surface of the sphere, $E_t = E_\theta = - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\Big|_{r=a} = 0$. This will determine the B_l. So far we have

$$\varphi(r, \theta) = - E_0 r\, P_1(\cos\theta) + \frac{B_0}{r} P_0(\cos\theta) + \frac{B_1}{r^2} P_1(\cos\theta) + \frac{B_2}{r^3} P_2(\cos\theta) +$$

$$= - E_0 r \cos\theta + \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3}\,\frac{1}{2}\left\{3\cos^2\theta - 1\right\} + \ldots$$

Then $E_t(r=a) = - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\Big|_{r=a} = - E_0 \sin\theta + \frac{B_1}{a^3}\sin\theta + \frac{B_2}{a^4}\,\frac{1}{2}\, 6\cos\theta\sin\theta + \ldots$

$$E_t(r=a) = 0 \ \text{ if } \ \frac{B_1}{a^3} = E_0 \quad \text{and} \quad B_i = 0 \quad \text{for } i > 1.$$

Thus

$$\varphi(r, \theta) = - E_0 r \cos\theta + \frac{a^3 E_0}{r^2}\cos\theta + \frac{B_0}{r}$$

The term $\frac{B_0}{r}$ is controlled by the charge on the sphere: $B_0 = Q$. For an uncharged sphere $B_0 = Q = 0$.

d. $\varphi(r, \theta) = - E_0 r \cos\theta + \frac{a^3 E_0}{r^2}\cos\theta + \frac{B_0}{r}$

The electric field is given by

$$\mathbf{E} = - \nabla\varphi = - \hat{\mathbf{r}}\frac{\partial\varphi}{\partial r} - \hat{\theta}\,\frac{1}{r}\frac{\partial\varphi}{\partial\theta}$$

For an uncharged sphere,

$$\mathbf{E} = \hat{\mathbf{r}} \left\{ E_0 \cos\theta + \frac{2\, a^3\, E_0}{r^3} \cos\theta \right\} + \hat{\theta} \left\{ \frac{- E_0\, r + \frac{a^3\, E_0}{r^2}}{r} \right\} \sin\theta$$

$$= \hat{\mathbf{r}}\ E_0 \cos\theta \left\{ 1 + \frac{2a^3}{r^3} \right\} + \hat{\theta}\ E_0 \sin\theta \left\{ - 1 + \frac{a^3}{r^3} \right\}$$

e. The charge density on the sphere is given by

$$\sigma = \frac{1}{4\pi} E_n = \frac{1}{4\pi} E_r(r=a) = \frac{3\ E_0 \cos\theta}{4\pi}$$

f. The total charge induced on the sphere is $Q = \int_0^\pi \sigma(\theta)\ 2\pi\, r\sin\theta\ r d\theta$ =

$$= \frac{3}{2} E_0\, a^2 \int_0^\pi \cos\theta \sin\theta\ d\theta = \frac{3}{2} E_0\, a^2 \left. \frac{\cos^2\theta}{2} \right|_0^\pi = 0$$

QED

Can also find potentials if the charge on a surface is specified. Use conditions that φ is continuous on the charged surface, and that $\mathbf{E}$ is discontinuous on the surface by $4\pi\sigma$.

Full spherical variation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left\{l(l+1) - \frac{m^2}{\sin^2\theta}\right\} P = 0$$

Generalized Legendre Equation, solutions are Associated Legendre Polynomials

Substitute $x = \cos\theta$

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left\{l(l+1) - \frac{m^2}{1-x^2}\right\} P = 0$$

As before, to have P finite at $x = \pm 1$, ($\theta = 0, \pi$), l must be an integer and m = integer between $\pm$ l.

Solution $P_l^m(x)$ can be found from differentiating Rodriguez formula

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_l(x)}{dx^m}$$

Orthogonality for fixed m:

$$\int_{-1}^{1} P_{l'}^m(x)\, P_l^m(x)\, dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l\,l'}$$

Orthogonality in m is taken care of by $Q(\varphi) = e^{\pm im\varphi}$. Thus the product

$P_l^m(\cos\theta)\, e^{\pm im\varphi}$ is orthonormal in both l and m.

If we collect together the angular parts of the solutions $P_l^m(\cos\theta)\, Q_m(\varphi)$, and make them orthonormal, we produce the *spherical harmonics*:

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}$$

For odd m, P_l^m changes sign if $m \rightarrow -m$: $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$

Thus $Y_{l,-m}(\theta,\varphi) = (-1)^m\, Y^*_{lm}(\theta,\varphi)$

Orthonormality:

$$\int Y^*_{l'm'}(\theta, \varphi)\, Y_{lm}(\theta, \varphi)\, d\Omega = \delta_{ll'}\, \delta_{mm'}$$

where $\int d\Omega = \int_0^{\pi} d\theta \int_0^{2\pi} \sin\theta\, d\varphi$

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{i\varphi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\, \cos\theta$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{-i\varphi}$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, \sin^2\theta\, e^{2i\varphi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\, \sin\theta \cos\theta\, e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

.
.
.

Note $Y_{l0}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$

Y_{lm} can be used to expand an arbitrary function:

$$g(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta. \varphi)$$

$$A_{lm} = \int Y^*_{lm}(\theta, \varphi)\, g(\theta, \varphi)\, d\Omega$$

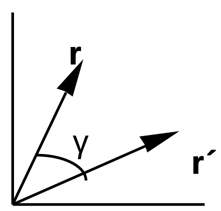
Full solution to Laplace's equation:

$$\varphi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{A_{lm} r^l + B_{lm} / r^{l+1}\right\} Y_{lm}(\theta, \varphi)$$

Coefficients A_{lm} and B_{lm} are determined by the boundary conditions

Interesting properties (which we will use later):

1. Addition theorem:



$$P_l(\cos\gamma) = \frac{4\pi}{2l + 1} \sum_{m=-l}^{l} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

• gives $\cos\gamma$ in terms of θ, φ and θ', φ' separately.

• if $\gamma \to 0$ then $\theta \to \theta'$ and $\varphi \to \varphi'$. Then

$$P_l(1) = 1 = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} |Y_{lm}(\theta, \varphi)|^2$$

"sum rule:" $$\sum_{m=-l}^{l} |Y_{lm}(\theta, \varphi)|^2 = \frac{2l+1}{4\pi}$$

2. Expansion of $\frac{1}{|\mathbf{r} - \mathbf{r}'|}$:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

$r_<$ is the lesser of r and r´, $r_>$ is the greater of the two.

• proved by making rotating axes so $\mathbf{r}'$ is along z to get azimuthal symmetry, then expanding in $P_l(\cos\theta)$. Jackson (3.38)

Use addition theorem:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

$$= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

This allows solutions to Poisson's equation by the same techniques as we have used to solve Laplace's equation:

1. Expand the integrals in $\oint_V d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$ as above and do the integrals term by term, as integrals of spherical harmonics.

2. Construct the Greens function as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \chi(\mathbf{r}, \mathbf{r}')$$

where both terms are expanded in spherical harmonics. Then adjust the coefficients to satisfy the boundary conditions on $G(\mathbf{r}, \mathbf{r}')$.

Connection to angular momentum in QM:

Angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \frac{h}{2\pi i} \nabla$

$$L^2 \psi = -\left(\frac{h}{2\pi}\right)^2 (\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla\psi) \qquad \psi = \text{wave function}$$

Vector identities (product rule, etc see Powell and Craseman, Quantum Mechanics, Addison Wesley, 1961, p 204)

$$L^2 \psi = -\left(\frac{h}{2\pi}\right)^2 r^2 \left\{ \nabla^2 - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) \right\}$$

In spherical polar coordinates,

$$L^2 \psi = -\left(\frac{h}{2\pi}\right)^2 \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2\psi}{\partial\varphi^2} \right\}$$

L^2 is the angular part of ∇^2 (times r^2)

This the same differential equation solved by $P(\theta)Q(\varphi) = Y_{lm}(\theta,\varphi)$

$$\frac{1}{Y_{lm}} L^2 Y_{lm} = \left(\frac{h}{2\pi}\right)^2 l(l + 1)$$

Thus Y_{lm} the eigenfunction of angular momentum

$$L^2\, Y_{lm} = l(l+1)\left(\frac{h}{2\pi}\right)^2 Y_{lm}$$

and

$$L_z\, Y_{lm} = m\,\frac{h}{2\pi}\, Y_{lm}$$

expansion in Y_{lm} is expansion in quanta of L^2 and L_z .

Cylindrical coordinates:

$$\frac{\partial^2\varphi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\varphi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\varphi}{d\varphi^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

Assume $\varphi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$

Substitute, divide by RQZ, get three terms each of one variable whose sum is zero. Each term must be a constant, sum of constants = 0.

Three equations:

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad\qquad Z(z) = e^{\pm kz}$$

$$\frac{d^2Q}{d\varphi^2} + \nu^2 Q = 0 \qquad\qquad Q(\varphi) = e^{\pm i\nu\varphi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0 \quad \text{Bessel's equation}$$

standard form: let $x = k\rho$

$$\rightarrow \qquad \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

Assume a power series solution

$$R(x) = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j$$

Find the recursion relation

$$a_{2j} = - \frac{1}{4j(j + \alpha)} a_{2j - 2}$$

etc.

Properties can be found in Jackson and elsewhere, we will not consider them further.

Multipole expansions of φ

- New method for solving electrostatic problems: calculation of each multipole of φ from the charge distribution
- Define dipole moments and get a feel for their properties- used everywhere in physics

Consider a charge distribution $\rho(\mathbf{r}')$ confined to some regions of space, and no boundaries:

$$\varphi(\mathbf{r}) = \oint_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

Expand $$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l + 1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

- $r > r'$ outside the region

$$\varphi(\mathbf{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \int_V \rho(\mathbf{r}')\, r'^{l}\, Y^*_{lm}(\theta',\varphi')\, d^3r'$$

Define $q_{lm} = \int_V \rho(\mathbf{r}')\, r'^{l}\, Y^*_{lm}(\theta',\varphi')\, d^3r'$

$$q_{00} = \sqrt{\frac{1}{4\pi}} \int_V \rho(\mathbf{r}')\, d^3r' = \sqrt{\frac{1}{4\pi}}\, Q \quad \text{(total charge)}$$

Monopole moment

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int_V \rho(\mathbf{r}')\, r'\sin\theta'\, e^{-i\varphi'}\, d^3r'$$

Note $\quad r'\sin\theta\, e^{-i\varphi} = r'\sin\theta' (\cos\varphi' - i\sin\varphi')$

$$= x' - i\,y'$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \left\{ \int_V \rho(\mathbf{r}')\, x' d^3r' - i \int_V \rho(\mathbf{r}')\, y' d^3r' \right\}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int_V \rho(\mathbf{r}')\, z' d^3r'$$

dipole moments

etc. for quadrupole moments, octopole moments

$$\varphi(\mathbf{r}) = 4\pi\, q_{00} \frac{Y_{00}}{r} + \frac{4\pi}{3} \left\{ \frac{q_{11} Y_{11} + q_{10} Y_{10} + q_{1\,-1} Y_{1\,-1}}{r^2} \right\} + \ldots$$

$$= \frac{Q}{r} \qquad \text{monopole term}$$

$$+ \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \qquad \text{dipole term} \qquad \mathbf{p} = \int_V \mathbf{r}' \rho(\mathbf{r}')\, d^3r'$$

$$+ \quad \text{quadrupole terms etc.}$$

Can also make the expansion in Cartesian coordinates:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

$$= \frac{1}{|\mathbf{r}|} + \frac{\partial}{\partial x'}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\Bigg|_{\mathbf{r}'=0} x' + \frac{\partial}{\partial y'}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\Bigg|_{\mathbf{r}'=0} y' + \frac{\partial}{\partial z'}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\Bigg|_{\mathbf{r}'=0} z'$$

$$+ \quad \frac{1}{2}\frac{\partial^2}{dx^2}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\Bigg|_{\mathbf{r}'=0} x'^2 + \ldots$$

$$= \frac{1}{|\mathbf{r}|} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \ldots$$

$$\varphi(\mathbf{r}) = \frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \ldots$$

Electric field $\mathbf{E}(\mathbf{r})$ for the potential :

monopole term: same as point charge at origin

$$\mathbf{E} = -\nabla \varphi(\mathbf{r}) = \frac{Q}{r^3}\mathbf{r}$$

dipole term: $\mathbf{E}(\mathbf{r}) = -\nabla\left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3}\right) = -\left(\mathbf{p} \cdot \mathbf{r}\, \nabla\left(\frac{1}{r^3}\right) + \frac{\nabla(\mathbf{p} \cdot \mathbf{r})}{r^3}\right)$

$$= \frac{3\,\mathbf{p} \cdot \mathbf{r}}{r^4}\hat{\mathbf{r}} - \frac{\mathbf{p}}{r^3}$$

$\nabla(\mathbf{p} \cdot \mathbf{r}) = \mathbf{p}$ by direct evaluation in Cartesian coordinates or by use of product rule

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}$$

• dipole field is same form for electric and magnetic

• dipole field closes on itself, unlike point charge, which extends to infinity

• any charge distribution can be replaced with its multipoles at the origin. If all multipoles are known, potential is precisely defined. Multipole expansion useful only when small number of terms is adequate, often only monopole and dipole.

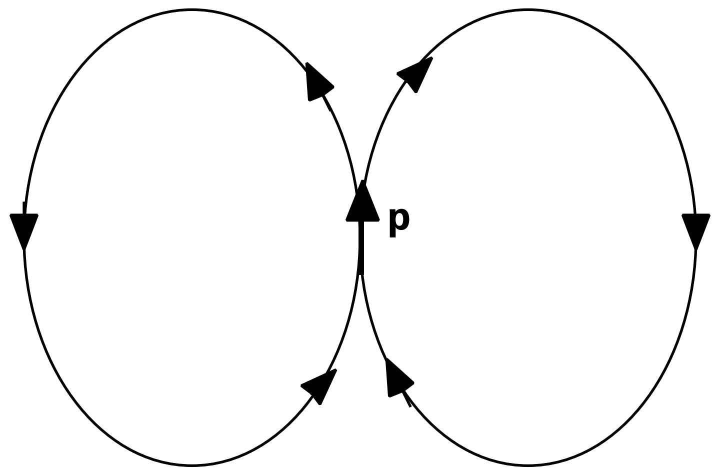
• We derived dipole as a property of a finite charge distribution. However, we can consider a "point dipole" like a point charge: no dimensions, located at a point, and acting as a source of electric field and potential.

point dipole at origin:	at $\mathbf{r}'$:
$\varphi(\mathbf{r}) = \dfrac{\mathbf{p} \cdot \mathbf{r}}{r^3}$	$\dfrac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{\lvert \mathbf{r} - \mathbf{r}' \rvert^3}$
$\mathbf{E}(\mathbf{r}) = \dfrac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} - \dfrac{\mathbf{p}}{r^3}$	$\dfrac{3(\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}'))(\mathbf{r} - \mathbf{r}')}{\lvert \mathbf{r} - \mathbf{r}' \rvert^5} - \dfrac{\mathbf{p}}{\lvert \mathbf{r} - \mathbf{r}' \rvert^3}$

• the monopole field falls off like $\frac{1}{r}$ the dipole like $\frac{1}{r^2}$ quadrupole like $\frac{1}{r^3}$ etc. Thus at large distances, all charge distributions look like monopoles (a point charge at the origin). Closer to the charge, the dipole term must be taken, even closer, the quadrupole term, etc.

• to get physical intuition about how point dipoles behave, think of a "physical dipole", two charges of opposite sign separated by **d**. That charge distribution has a dipole moment q**d**, but also quadrupole moments and higher moments. It is not a pure dipole. See Griffiths, prob. 3.30. Nevertheless, it helps to indicate direction of dipole (from negative to positive side), and the fields produced, and the response of a dipole to an external field.

dipole field:



E at special points:

$\mathbf{r} \parallel \mathbf{p}$: $\mathbf{p} \cdot \mathbf{r} = pr$ $\quad \mathbf{E} = \frac{3p}{r^3}\hat{\mathbf{r}} - \frac{p}{r^3}\hat{\mathbf{r}} = \frac{2p}{r^3}\hat{\mathbf{r}}$

r perpendicular to **p** : $\mathbf{E} = -\frac{\mathbf{p}}{r^3}$

Electrostatic Energy

We will show that the energy needed to assemble a charge distribution is

$$W = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \qquad \text{points}$$

$$= \frac{1}{2} \int_V d^3r \int_V d^3r' \frac{\rho(\mathbf{r})\, \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{continuous charges}$$

$$= \frac{1}{2} \int_V \rho(\mathbf{r})\, \varphi(\mathbf{r})\, d^3r$$

$$= \frac{1}{8\pi} \int_V |\mathbf{E}|^2\, d^3r$$

Consider a point charge q_1 at the origin.

$$\mathbf{E}(\mathbf{r}) = \frac{q_1}{r^3}\, \mathbf{r}$$

Energy to bring a second point charge q_2 up from infinity:

$$W = \int_\infty^{\mathbf{r}_0} \mathbf{F} \cdot \mathbf{dl} = \int_\infty^{\mathbf{r}_0} \frac{q_1 q_2}{r^3}\, \underbrace{\mathbf{r} \cdot \mathbf{dr}}$$

↑
$-\,r dr$ because $\mathbf{dr}$ points opposite to $\mathbf{r}$

$$= -\int_\infty^{r_0} \frac{q_1 q_2}{r^2}\, dr = q_1 q_2 \left. \frac{1}{r} \right|_\infty^{r_0} = \frac{q_1 q_2}{r_0}$$

$$\rightarrow \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \quad \text{for } q_1 \text{ at } \mathbf{r}_1 \text{ instead of origin}$$

3^{rd} charge: $$W = \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{q_1 q_3}{|\mathbf{r}_1 - \mathbf{r}_3|} + \frac{q_2 q_3}{|\mathbf{r}_2 - \mathbf{r}_3|}$$

• self energy terms excluded, each charge paired with each other charge once.

$$W = \sum_{i>j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Proves first point.

Generalize to continuous charges via superposition:

$$W = \underset{\substack{\uparrow \\ \text{avoids double counting}}}{\frac{1}{2}} \int_V d^3r \int_V d^3r' \frac{\rho(\mathbf{r})\, \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Note self-energy included, when $\mathbf{r} = \mathbf{r}'$.

Proves second point.

Operate on this expression to get the other two:

note $$\varphi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

$$W = \frac{1}{2} \int_V d^3r\, \rho(\mathbf{r})\, \varphi(\mathbf{r})$$

note $$\rho(\mathbf{r}) = \frac{1}{4\pi} \nabla \cdot \mathbf{E}$$

$$W = \frac{1}{2} \int_V d^3r \, \frac{1}{4\pi} \, \nabla \cdot \mathbf{E} \, \varphi(\mathbf{r})$$

Integrate by parts, through away term at infinity where all fields and potentials are zero

$$W = -\frac{1}{8\pi} \int_V \mathbf{E} \cdot \nabla\varphi \, d^3r = \frac{1}{8\pi} \int_V |\mathbf{E}|^2 \, d^3r$$

• Emphasis can be put on the charges themselves in first expression, or on a property of space itself as in third expression.

• Above applies when all fields are produced by charges q or $\rho(\mathbf{r})$. If we are interested in the energy of a charge distribution $\rho(\mathbf{r})$ with a external field, not produced by the charge distribution, then double counting the energy does not arise. Energy required to bring a point charge q from ∞ to $\mathbf{r}_0$ in presence of external field $\mathbf{E}_{ext}(\mathbf{r})$:

$$W = \int q\mathbf{E}_{ext} \cdot \mathbf{dl} = q \int_{\infty}^{\mathbf{r}_0} -\nabla\varphi_{ext} \cdot \mathbf{dl} = -q \int_{\infty}^{\mathbf{r}_0} \frac{\partial\varphi_{ext}}{\partial r} \, \hat{\mathbf{r}} \cdot \mathbf{dl}$$

↑

dr, dl point in opposite directions

$$= q \left\{\varphi_{ext}(\mathbf{r}_0) - \varphi_{ext}(\infty)\right\} = q \, \varphi_{ext}(\mathbf{r}_0)$$

For a continuous charge distribution add the energy required for all elements $\rho(\mathbf{r})$ by integration:

$$W = \int_V \rho(\mathbf{r}) \, \varphi_{ext}(\mathbf{r}) \, d^3r$$

We are not interested in the energy required to assemble the charge against other parts of the charge distribution, so there is no need to keep track of the interactions within the charge distribution, and no $\frac{1}{2}$. In this expression, $\varphi(\mathbf{r})$ refers only to the external potential, not to the potential due to $\rho(\mathbf{r})$.

Multipole expansion of the energy

Consider a charge distribution $\rho(\mathbf{r})$ in an *external* potential. $\left(\text{No factor of } \frac{1}{2}.\right)$

$$W = \int_V \rho(\mathbf{r})\, \varphi(\mathbf{r})\, d^3r$$

Expand the potential about some origin:

$$\varphi(\mathbf{r}) = \varphi(0) + \nabla\varphi\Big|_0 \cdot \mathbf{r}, + \frac{1}{2}\frac{\partial^2 \varphi}{\partial r_i \partial r_j}\Big|_0 r_i r_j + \ldots$$

$$W = \underset{\substack{\uparrow \\ \text{monopole}}}{Q\,\varphi(0)} + \underset{\substack{\uparrow \\ \text{dipole}}}{\nabla\varphi\Big|_0 \cdot \mathbf{p}} + \underset{\substack{\uparrow \\ \text{quadrupole}}}{\frac{1}{2}\frac{\partial^2 \varphi}{\partial r_i \partial r_j}\Big|_0 Q_{ij}} + \underset{\text{etc}}{\ldots}$$

Shows how the multipoles interact with external field.

$$W = Q\,\varphi(0) - \mathbf{E} \cdot \mathbf{p} + \ldots$$

A uniform external field produces no force on the dipole, but does produce a torque. A non-uniform field produces a force also.

$$\mathbf{F} = -\nabla W = -Q\,\mathbf{E} + \nabla(\mathbf{p} \cdot \mathbf{E})$$

$$\nabla (\mathbf{p} \cdot \mathbf{E}) = \underset{\substack{\uparrow \\ 0}}{\mathbf{p} \times (\nabla \times \mathbf{E})} + \underset{\substack{\uparrow \\ 0}}{\mathbf{E} \times (\nabla \times \mathbf{p})} + (\mathbf{p} \cdot \nabla)\, \mathbf{E} + \underset{\substack{\uparrow \\ 0}}{(\mathbf{E} \cdot \nabla)\, \mathbf{p}}$$

$$= (\mathbf{p} \cdot \nabla)\, \mathbf{E} = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} \right) \mathbf{E}$$

$$\tau = -\frac{\partial W}{\partial \theta} = pE \sin\theta = |\mathbf{p} \times \mathbf{E}| \qquad \tau = \mathbf{p} \times \mathbf{E}$$

Interaction energy of point charge and dipole:

$\mathbf{E}$ due to point charge: $\frac{q}{r^3}\, \mathbf{r}$

$W = -\, \mathbf{p} \cdot \mathbf{E} = -\frac{q}{r^3}\, \mathbf{p} \cdot \mathbf{r}$

φ due to $\mathbf{p}$: $\frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$

$W = q\, \varphi(-\, \mathbf{r}) = -\, q\, \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$ same, as it must be.

Dielectric media

Thus far, all fields have been due to free charges. Point charge and superposition have been the basis for all our work. Now we see that a dipole moment also produces fields and potentials at distant points, even though its total charge is zero. We have no method for finding these effects. We will develop a simple method.

Physical situation: Two kinds of dipoles.

1. Fixed charges, like in H_2O, which are separated by a fixed distance. Usually they can rotate, and line up with a field, so their average moment increases in presence of field.

2. Free charges confined to a volume or surface, like the uncharged conducting sphere. In $\mathbf{E}$, a charge is induced which can increase or decrease its polarization in response to the field strength.

Both dipoles change their strength in response to the applied field.

There are also higher moments, but they are usually small compared to the dipole. Recall the higher moments fall off faster with distance than the dipole.

Modify Maxwell's equations to include dielectric.

Define $\mathbf{P}(\mathbf{r})$ = density of dipoles at $\mathbf{r}$, i. e., assume there is a distribution of point dipoles in space. In the volume element d^3r, add up the vector sum of all the dipoles, $\sum_i \mathbf{p}_i$. Set this equal to $\mathbf{P}(\mathbf{r})\, d^3r$:

$$\mathbf{P}(\mathbf{r})\, d^3r = \sum_i \mathbf{p}_i$$

$\mathbf{P}(\mathbf{r})$ = density of dipole moment/unit volume or "polarization"

Terms are identical.

$\mathbf{P}(\mathbf{r})$ plays same role as charge density $\rho(\mathbf{r})$. $\rho(\mathbf{r})$ gives density of charges at $\mathbf{r}$, $\mathbf{P}(\mathbf{r})$ give density of dipole moment at $\mathbf{r}$. This is source term, whose potential we want to calculate. It is a vector, not a scalar, but otherwise we treat it the same.

For point dipole at origin: $\varphi(\mathbf{r}) = \dfrac{\mathbf{p} \cdot \mathbf{r}}{r^3}$

at point $\mathbf{r}'$: $\varphi(\mathbf{r}) = \dfrac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = -\mathbf{p} \cdot \nabla \left(\dfrac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$

For continuous dipole distribution use superposition, as we did for charges:

$$\varphi(\mathbf{r}) = \int_V \frac{\mathbf{P}(\mathbf{r}')\, d^3r' \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = -\int_V \mathbf{P}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3r'$$

Note $\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$ prove by differentiation in Cartesian coordinates. Reason is that, e. g., if $\mathbf{r}$ and $\mathbf{r}'$ are on same line, then $|\mathbf{r}-\mathbf{r}'|$ gets larger as $\mathbf{r}$ increases, but gets smaller if $\mathbf{r}'$ increases.

$$\varphi(\mathbf{r}) = +\int_V \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r'$$

Integrate by parts, keeping surface term:

$$\varphi(\mathbf{r}) = -\int_V \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

Note the integration by parts can be understood as application of product rule and divergence theorem:

$$\nabla\cdot(f\,\mathbf{A}) = f\,\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla f$$

$$\nabla\cdot\left(\frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|}\right) = \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\nabla\cdot\mathbf{P} + \mathbf{P}\cdot\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$\int_V \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3r' = \int_V \left[\nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] d^3r'$$

Divergence theorem: $\displaystyle\int_V \nabla'\cdot\left(\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) = \int_S \frac{\mathbf{P}\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}\, da'$

$$\varphi(\mathbf{r}) = \int_V \mathbf{P}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3r' = \int_S \frac{\mathbf{P} \cdot \mathbf{n}}{|\mathbf{r} - \mathbf{r}'|} \, da' + \int_V \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'$$

This looks identical to φ due to a volume charge $\rho(\mathbf{r}') = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$ and a surface charge $\sigma = \mathbf{P} \cdot \mathbf{n}$

Define $\rho_{equiv} = \rho_{bound} = \rho_{pol} = -\nabla \cdot \mathbf{P}(\mathbf{r})$

$\sigma_{equiv} = \sigma_{bound} = \sigma_{pol} = \mathbf{P} \cdot \mathbf{n}$

Now we can calculate fields and potentials due to a given distribution of dipoles $\mathbf{P}(\mathbf{r})$, by substituting the equivalent charge distributions and using the usual techniques.

• Every polarization is *equivalent* to a volume and surface charge distribution as above. Same will be true of magnetization, equivalent to a current distribution.

Example: Calculate the field from a slab of uniformly polarized material situated between $z = \pm z_0$, where $\mathbf{P} = P\mathbf{k}$.

$\rho_{pol} = -\nabla \cdot \mathbf{P} = 0$

$\sigma_{pol} = \mathbf{P} \cdot \mathbf{n} = +P$ at $z = z_0$ $\quad -P$ at $z = -z_0$

By Gauss' Law, from the surface at z_0, $\quad \mathbf{E} = 2\pi\sigma\mathbf{k} = 2\pi\mathbf{P} \quad z > z_0$

$\mathbf{E} = -2\pi\sigma\mathbf{k} = -2\pi\mathbf{P} \quad z < z_0$

from the surface at $-z_0$ $\quad \mathbf{E} = 2\pi\sigma\mathbf{k} = -2\pi\mathbf{P} \quad z > -z_0$

$\mathbf{E} = -2\pi\sigma\mathbf{k} = 2\pi\mathbf{P} \quad z < z_0$

The sum of the two is $\mathbf{E} = 0$ for $|z| > z_0$, $\quad \mathbf{E} = -4\pi\mathbf{P}$ for $|z| < z_0$

ρ_{pol} produces exactly the same potential and field as an equivalent distribution of free charges. Same everything, and all the equations go through as before if ρ is replaced by $\rho_{free} + \rho_{pol}$.

$$-\nabla^2\varphi = \nabla \cdot \mathbf{E} = 4\pi\rho_{free} + 4\pi\rho_{pol}$$

$$= 4\pi\rho_{free} - 4\pi\,\nabla \cdot \mathbf{P}$$

For convenience, define $\mathbf{D} = \mathbf{E} + 4\pi\,\mathbf{P}$

Then $$\nabla \cdot \mathbf{D} = \nabla \cdot \left(\mathbf{E} + 4\pi\mathbf{P}\right) = 4\pi\,\rho_{free}$$

$\mathbf{D}$ is the field of free charges

$\mathbf{E}$ is the field of all charges

First Maxwell equation, modified for dielectrics.

• We seem to be ignoring the surface term $\mathbf{P} \cdot \mathbf{n}$. However, it is contained in $\nabla \cdot \mathbf{P}$: if $\mathbf{P}$ suddenly goes to zero, $\nabla \cdot \mathbf{P}$ has a sharp discontinuity which is the surface term.

Example: uniformly polarized half space z > 0, with $\mathbf{P} = P\ \mathbf{k}$.

$$\mathbf{P}(\mathbf{r}) = P\ \mathbf{k}\ \theta(z) \quad = P\ \mathbf{k} \qquad \text{if } z > 0$$

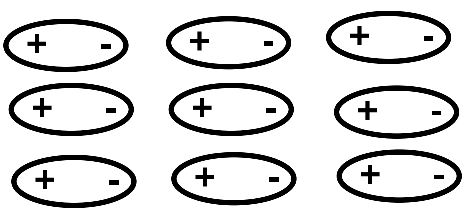
$$= 0 \qquad \text{if } z < 0$$

$$\nabla \cdot \mathbf{P} = \frac{\partial P_z}{\partial z} = P\ \delta(z) \qquad \oint_V \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\ d^3r' = \int \frac{-P}{|\mathbf{r} - \mathbf{r}'|}\,\delta(z)\ dx\ dy\ dz$$

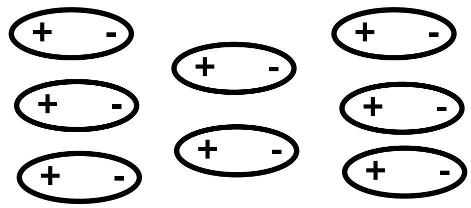
$= \int_S \frac{\mathbf{P} \cdot \mathbf{n}}{|\mathbf{r} - \mathbf{r}'|}\, da$ where S is the xy plane, $\mathbf{n}$ = outward drawn normal = - $\mathbf{k}$

• Discontinuities in **P** do not occur in the real world, so surface terms do not arise in physical problems. However, it is useful to idealize certain problems, like uniformly polarized objects.

• Interpret $-\nabla \cdot \mathbf{P}$ and $\mathbf{P} \cdot \mathbf{n}$ as uncompensated charge.



Uniform density $\mathbf{P}(\mathbf{r})$



non uniform density $\nabla \cdot \mathbf{P} \neq 0$

Two ways to get effective polarization charge: **P** varies in space, or **P** abruptly ends, at a surface. Then there is an effective surface charge $\sigma_{pol} = \mathbf{P} \cdot \mathbf{n}$.
This is the surface term we threw away in the integration by parts above.

This simplifies the problem. Given a charge density $\rho(\mathbf{r})$ and a dipole moment density $\mathbf{P}(\mathbf{r})$, the potentials and fields can be calculated from

$$-\nabla^2 \varphi = \nabla \cdot \mathbf{E} = 4\pi\,(\rho_{free} + \rho_{pol}) = 4\pi\,(\rho_{free} - \nabla \cdot \mathbf{P})$$

Here, the terms in **P** appear on right as they should if they are sources of fields. Homework will treat cases of permanent polarization, solved by replacing $\mathbf{P}(\mathbf{r})$ with its equivalent charge distribution.

But, the most common cases have **P** as a response to **E**, ie, the dipole moment density is induced by the field. Example: polar molecules. Then **P** depends on **E**, and the problem must be iterated infinitely to get a solution.

Simplification: Assume $\mathbf{P} = \chi\mathbf{E}$, ie, linear relation. Then **P** can be considered a response to **E**, created by the source charge $\rho(\mathbf{r})$. Makes sense to put **P** on LHS as a response

$$\nabla \cdot (\mathbf{E} + 4\pi\mathbf{P}) = \nabla \cdot \mathbf{D} = 4\pi\,\rho_{free}$$

If $\mathbf{P} = \chi\mathbf{E}$ then $\mathbf{D} = \mathbf{E} + 4\pi\chi\mathbf{E} = (1 + 4\pi\chi)\,\mathbf{E}$

Define $1 + 4\pi\chi = \kappa$, the dielectric constant

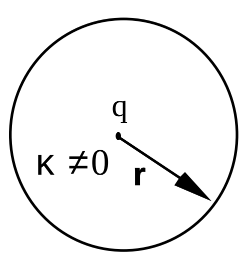
Note that φ and **E** retain their former meanings, $\mathbf{E} = -\nabla\varphi$ and $\mathbf{F} = q\,\mathbf{E}$.

Thus $\nabla \times \mathbf{E} = 0$. Note $\nabla \times \mathbf{D} = \nabla \times (\kappa\mathbf{E}) = \kappa\,\nabla \times \mathbf{E} + \nabla\kappa \times \mathbf{E} = \nabla\kappa \times \mathbf{E}$

If κ varies in space, $\nabla \times \mathbf{D} \neq 0$.

For all dielectrics we will consider, κ is constant in space and $\nabla \times \mathbf{D} = 0$, except at boundaries, where κ changes discontinuously.

Effect of dielectric: point charge in dielectric medium.



Gauss law for **D** : $\nabla \cdot \mathbf{D} = 4\pi \rho_{free}$

$$\int_S \mathbf{D} \cdot \mathbf{n}\, da = 4\pi q$$

$$D_n\, 4\pi r^2 = 4\pi q$$

$$D_n = \frac{q}{r^2}$$

$$\mathbf{D} = \kappa \mathbf{E} \qquad E_n = \frac{1}{\kappa}\frac{q}{r^2}$$

$$\text{and } \mathbf{P} = \frac{\mathbf{D} - \mathbf{E}}{4\pi} = \frac{\kappa - 1}{4\pi}\mathbf{E} = \frac{\kappa - 1}{4\pi\kappa}\mathbf{D} \qquad P_n = \frac{\kappa - 1}{4\pi\kappa}\frac{q}{r^2}$$

The dielectric reduces the **E** field by κ. The reduction comes from the polarization of the dipoles in the dielectric.

Note $\rho_{pol} = -\nabla \cdot \mathbf{P} = -\frac{\kappa - 1}{4\pi\kappa}\nabla \cdot \mathbf{D} = -\frac{\kappa - 1}{4\pi\kappa}4\pi\rho = -\frac{\kappa - 1}{\kappa}q\delta(\mathbf{r})$

Thus even though **P** varies in space everywhere, $\rho_{pol} = -\nabla \cdot \mathbf{P}$ is zero except at the origin. This is *required* by linearity of **P**, **E**, and **D**,

$$\rho_{pol} = -\nabla \cdot \mathbf{P} = -\frac{\kappa - 1}{\kappa}\rho(\mathbf{r})$$

Boundary Conditions on D and E

Consider a boundary between two dielectrics κ_1 and κ_2 .

BC for **E** : draw a rectangular contour spanning the boundary. Use curl theorem:

$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \int_C \mathbf{A} \cdot d\mathbf{l}$ where S is the surface contained by C.

$$\nabla \times \mathbf{E} = 0 \qquad 0 = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{a} = \int_C \mathbf{E} \cdot d\mathbf{l}$$

Shrink the contour perpendicular to surface, so that $E_t^1 L - E_t^2 L = 0$

E_t is continuous across the boundary

BC for **D** : Draw a Gaussian pillbox, use Gauss' Law for **D** and divergence theorem:

$$\nabla \cdot \mathbf{D} = 4\pi \rho_{free} \qquad \int_S \mathbf{D} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{D}\, d^3r = 4\pi \text{ charge enclosed}$$

Shrink the box perpendicular to surface $\left(D_n^1 - D_n^2\right) A = 4\pi \sigma_{free} A$

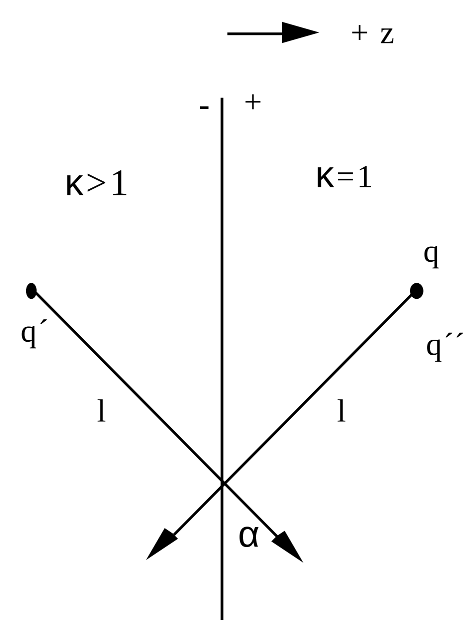
$$\left(D_n^1 - D_n^2\right) = 4\pi \sigma_{free}$$

i. e., normal component of **D** is discontinuous by amount of free surface charge. This is not related to bound surface charge $\mathbf{P} \cdot \mathbf{n}$.

• At boundary between two dielectrics, where **P** changes discontinuously, **D** is continuous (assuming no free surface charges). Then

$E_{n1} + 4\pi P_{n1} = E_{n2} + 4\pi P_{n2}$. If P_n changes by ΔP, E_n changes by $- 4\pi \Delta P$. This jump in E is equivalent to a surface charge $\sigma = - \Delta P = \mathbf{P} \cdot \mathbf{n}$. Thus the bound surface charge $\mathbf{P} \cdot \mathbf{n}$ is implicit in the BC.

Example problem:



Point charge in vacuum near boundary with dielectric.

Use method of images: Replace the boundary with an image charge, but keep the dielectric in place.

$$z > 0: \quad \varphi(\mathbf{r}) = \frac{q}{|\mathbf{r} - \mathbf{r}_q|} + \frac{q'}{|\mathbf{r} - \mathbf{r}_{q'}|} \qquad \kappa = 1$$

$$z < 0: \quad \varphi(\mathbf{r}) = \frac{1}{\kappa}\frac{q''}{|\mathbf{r} - \mathbf{r}_{q''}|} \qquad \kappa > 1$$

• The prefactor $\frac{1}{\kappa}$ is convenient to make $\mathbf{E} = -\nabla\varphi$ look like a point charge in a dielectric, and $\mathbf{D} = \kappa\,\mathbf{E}$ satisfy the Poisson Eq. for the point charge q''. We could define φ without the prefactor and we would get different equations for q's, but same φ and $\mathbf{E}$. This is Jackson's convention, and we follow it to avoid confusion.

$$z = 0: \quad E_t^+ = D_t^+ = \frac{q}{l^2}\cos\alpha + \frac{q'}{l^2}\cos\alpha = \left(q + q'\right)\frac{\cos\alpha}{l^2}$$

$$E_t^- = \frac{1}{\kappa} D_t^- = \frac{1}{\kappa}\frac{q''}{l^2}\cos\alpha$$

$$E_t^+ = E_t^- \qquad q + q' = \frac{q''}{\kappa}$$

$$D_n^+ = -\frac{q}{l^2}\sin\alpha + \frac{q'}{l^2}\sin\alpha = \left(q' - q\right)\frac{\sin\alpha}{l^2}$$

$$D_n^- = -\frac{q''}{l^2}\sin\alpha$$

$$D_n^+ = D_n^- \qquad q' - q = -q''$$

2 equations in 2 unknowns: $q - q' = q''$

$$q + q' = \frac{q''}{\kappa}$$

Add and subtract:

$$2q = q''\left(1 + \frac{1}{\kappa}\right)$$

$$2q' = q''\left(\frac{1}{\kappa} - 1\right)$$

$$q'' = \frac{2}{\left(1 + \frac{1}{\kappa}\right)}\, q = \frac{2\kappa}{1+\kappa}\, q$$

$$q' = \frac{1}{2}\,\frac{2\kappa}{1+\kappa}\left(\frac{1}{\kappa} - 1\right) q = \frac{1-\kappa}{1+\kappa}\, q$$

Homework: Substitute these values of q', q'' into $\varphi(\mathbf{r})$ and find the electric fields in all space and the bound volume and surface charges $-\nabla \cdot \mathbf{P}$ and $\mathbf{P} \cdot \mathbf{n}$.

Energy in Dielectrics

In presence of dielectric, $\mathbf{E}$ is reduced $\mathbf{F} = q\,\mathbf{E}$ is reduced, so energy to assemble the charge distribution is reduced.

Point charges in dielectric:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{\kappa}\,\frac{q_1}{|\mathbf{r} - \mathbf{r}_1|^3}\left(\mathbf{r} - \mathbf{r}_1\right)$$

$$W = \int_{\infty}^{\mathbf{r}_2} q\,\mathbf{E} \cdot d\mathbf{l} \qquad \text{as before}$$

$$W = \frac{1}{\kappa} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

For many point charges, $$W = \frac{1}{2} \sum_{i \neq j} \frac{1}{\kappa} \frac{q_1 q_2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Generalize to continuous charges: $$W = \frac{1}{2} \int_V \frac{1}{\kappa} \frac{\rho_{free}(\mathbf{r}') \, \rho_{free}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \, d^3r \, d^3r'$$

- Written in terms of free charges. Effect of dielectric is κ.

Recall $\nabla \cdot \mathbf{D} = 4\pi \rho_{free}$ and $\mathbf{D} = \kappa \mathbf{E}$ for linear dielectric

$$\nabla \cdot \mathbf{E} = -\nabla^2 \varphi = 4\pi \frac{\rho_{free}}{\kappa} \qquad \varphi(\mathbf{r}) = \int_V \frac{1}{\kappa} \frac{\rho_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'$$

$$W = \frac{1}{2} \int_V \varphi(\mathbf{r}) \, \rho_{free}(\mathbf{r}) \, d^3r$$ same as without dielectric

$$\rho_{free} = \frac{\nabla \cdot \mathbf{D}}{4\pi} \qquad W = \frac{1}{8\pi} \int_V \varphi(\mathbf{r}) \, \nabla \cdot \mathbf{D}(\mathbf{r}) \, d^3r$$

Integrate by parts, i. e., use the product rule for $\nabla \cdot (\varphi \mathbf{D})$ and the Divergence Theorem, throwing away the term at ∞.

$$W = \frac{1}{8\pi} \int_V -\nabla\varphi \cdot \mathbf{D} \, d^3r = \frac{1}{8\pi} \int_V \mathbf{E} \cdot \mathbf{D} \, d^3r$$

• We proved this for $\kappa \neq \kappa(\mathbf{r})$, but it is also true for $\kappa = \kappa(\mathbf{r})$, as shown by de Bartolo.

• This expression assumes linearity of **D** and **E**, and does not apply if the medium is non-linear.

End of electrostatics. Topics not covered:

Capacitance: For set of conductors, increasing Q on conductors increases **E** increases the potential difference between conducting surfaces.

Define $C = \dfrac{Q}{\Delta\varphi}$ where $\Delta\varphi$ is the potential difference between conducting surfaces containing Q charge.

Example: parallel plates with surface charge σ

Gauss law $E = 2\pi\sigma$ on opposite sides of each surface. Then between surfaces, $E = 4\pi\sigma$. $\Delta\varphi = \int E_n \, dl = 4\pi\sigma d$ $\quad C = \dfrac{Q}{\Delta\varphi} = \dfrac{\sigma A}{4\pi\sigma d} = \dfrac{A}{4\pi d}$

If the capacitor is filled with a dielectric, E is reduced by κ. Then

$\Delta\varphi = \dfrac{E}{\kappa} d = \dfrac{4\pi\sigma d}{\kappa}$ $\qquad C = \kappa \dfrac{A}{4\pi d}$

Maxwell stress tensor: transform volume forces to surface forces. Useful in treating electromagnetic phenomena

volume force $\mathbf{F} = q\,\mathbf{E} \rightarrow \int_V \rho(\mathbf{r})\,\mathbf{E}(\mathbf{r})\,d^3r$

Transform to surface force, using divergence theorem on each component.

$$\rho = \frac{\nabla \cdot \mathbf{E}}{4\pi} \qquad \mathbf{F} = \frac{1}{4\pi}\int_V (\nabla \cdot \mathbf{E})\, \mathbf{E}\, d^3r \quad = \frac{1}{4\pi}\int_V \left(\sum_{i=1}^{3} \frac{\partial E_i}{\partial r_i} \right) \mathbf{E}\, d^3r$$

$$F_k = \frac{1}{4\pi}\int_V \left(\sum_{i=1}^{3} \frac{\partial E_i}{\partial r_i} \right) E_k\, d^3r$$

Vector identities $= \dfrac{1}{4\pi}\displaystyle\int_V \sum_{i=1}^{3} \frac{\partial}{\partial r_i} \left(E_i E_k - \frac{1}{2}\delta_{ik} E^2 \right) d^3r$

define $T_{ik} = E_i E_k - \frac{1}{2}\delta_{ik} E^2$ = Maxwell stress tensor

For each k, $\displaystyle\sum_{i=1}^{3} \frac{\partial}{\partial r_i} T_{ik}$ looks like a divergence on the first index. Use the divergence theorem,

$$F_k = \frac{1}{4\pi}\int_V \sum_{i=1}^{3} \frac{\partial}{\partial r_i} T_{ik}\, d^3r = \frac{1}{4\pi}\oint_S \sum_{i=1}^{3} n_i T_{ik}\, d^2r$$

Transforms volume force into surface force, each component is surface integral of a tensor.

Stationery Currents

What happens when charges move? We will see that

- Equation of continuity relates $\mathbf{J}$ and ρ.

- Relation between **J** and **E**, Joule heating
- Approach to equilibrium.
- Equilibrium is equivalent to $\nabla \cdot \mathbf{J} = 0$.
- Current distribution $\mathbf{J}(\mathbf{r})$ is boundary value problem in electrostatics

Define **J** : due to moving charges n point charges/unit volume, moving with velocity **v**.

Cross section A. In Δt, all charges move by $\mathbf{v}\,\Delta t$. All charges in a box of volume $v\,\Delta t\,A$ pass through the end face. Charge = $n\,q\,v\,\Delta t\,A$

$$I = \frac{\text{charge}}{\text{unit time}} = \frac{n\,q\,v\,\Delta t\,A}{\Delta t} = n\,q\,v\,A = \text{current}$$

$$\mathbf{J} = \text{current density} = \frac{I}{A} = q\,n\,\mathbf{v}$$

Continuous charges : $\mathbf{J} = \rho\,\mathbf{v}$

Continuity equation

Consider volume V, surface S. Charge leaving V/unit time = $\int_S \mathbf{J} \cdot \mathbf{n}\, d^2r$

- This accounts for both entering and leaving charge

Total charge in S is reduced by amount leaving:

$$-\frac{\partial}{\partial t}\int_V \rho(\mathbf{r})\, d^3r = \int_S \mathbf{J} \cdot \mathbf{n}\, d^2r \underset{\substack{\uparrow \\ \text{divergence theorem}}}{=} \int_V \nabla \cdot \mathbf{J}\, d^3r$$

Volume of integration arbitrary $-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J}$ Continuity equation

- In steady state ("stationary currents") $\frac{\partial \rho}{\partial t} = 0$ $\quad \nabla \cdot \mathbf{J} = 0$.

Relation of **J** to **E**

Charges respond to forces, described by **E**. Expect $\mathbf{J} = f(\mathbf{E})$.

experiment: $\mathbf{J} = \sigma \mathbf{E}$ only linear term needed.

Surprising result! From Newton's laws,

$$\mathbf{F} = q\,\mathbf{E} = m\,\mathbf{a} = m\,\frac{\partial \mathbf{v}}{\partial t}$$

$$\mathbf{J} = n\,q\,\mathbf{v} \qquad \frac{\partial \mathbf{J}}{\partial t} = n\,q\,\frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{\partial \mathbf{J}}{\partial t} = n\,q\,\frac{q}{m}\,\mathbf{E} = \frac{nq^2}{m}\,\mathbf{E}$$

For constant $\mathbf{E}$, $\mathbf{J} = \frac{nq^2}{m}\mathbf{E}\,t$ increases without limit

Collisions prevent this, limiting time t for acceleration.

There is one case where there are no collisions and $\frac{\partial \mathbf{J}}{\partial t} = \frac{nq^2}{m}\mathbf{E}$:

superconductors. Leads to interesting new behavior.

Joule heating: $\mathbf{E}$ does work on ρ

$dW = \mathbf{F} \cdot d\mathbf{l}$

$$\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = \rho\,\mathbf{E} \cdot \frac{\mathbf{J}}{\rho} = \mathbf{E} \cdot \mathbf{J} = \frac{\mathbf{J} \cdot \mathbf{J}}{\sigma} \rightarrow I^2 R$$

- For superconductor, $\sigma = 0$ there are no losses

Approach to equilibrium

What happens if **E** is suddenly changed? Charges flow in response to **E**, perhaps changing **E**, until steady state is reached where ρ, **E**, and **J** are all constant in time.

Differential equation governing approach to steady state:

$$-\frac{\partial\rho}{\partial t} = \nabla\cdot\mathbf{J} = \nabla\cdot\sigma\,\mathbf{E} = \sigma\,\nabla\cdot\frac{\mathbf{D}}{\kappa} = \frac{\sigma}{\kappa}\,4\pi\,\rho$$

$$\rho(\mathbf{r},\,t) = \rho_0\,e^{-\frac{4\pi\sigma}{\kappa}t} + \text{const.}$$

↑ ↑

transient steady state

$$\frac{\kappa}{4\pi\sigma} = \text{time constant of system} = \text{time to reach equilibrium}$$

Usually very fast, faster the larger σ

Current distribution in conductor

Note in presence of **J**, $\mathbf{E} \neq 0$ in conductor.

Problem: given conductor of arbitrary shape, what is **J**(**r**) ?

Solve with differential equation and boundary conditions

$$\mathbf{J} = \sigma\,\mathbf{E} = -\sigma\,\nabla\varphi$$

$$\nabla\cdot\mathbf{J} = 0 \qquad\qquad -\sigma\ {}^2\varphi = 0$$

φ satisfies Laplace's equation, and $\mathbf{J} = -\sigma\,\nabla\varphi$

BC corresponds to physical situation. Often battery is used to apply $V = \Delta\varphi$ along length of wire, each cross section is at constant potential.

Then $$\varphi(x, y, z) = \varphi_0 - E z$$

$$\mathbf{J}(x, y, z) = -\sigma\nabla\varphi = \sigma E \mathbf{k}$$

ie, **J** is uniform over the cross section of the wire and constant along its length.

Magnetostatics

Old subject:
- lodestones known to Greeks
- compasses used by mariners long before they were understood

Harder to visualize than electrostatics

Coulomb began to publish on electrostatics in 1785, law given before 1800

Oersted: 1819 Noticed B due to moving charge

Ampere: 1820-25 thorough set of expts on forces between circuits

Biot and Savart: Force between two long straight wires

Magnetostatics deals with

B = magnetic induction field for free space

H = magnetic field for magnetized material.

B is called "induction field" because it is the field which causes induced currents in Faraday's law.

Historically, **H** was dealt with before **B**, so **H** got the general name "magnetic field"

Basic difference with electrostatics:

- Vector character

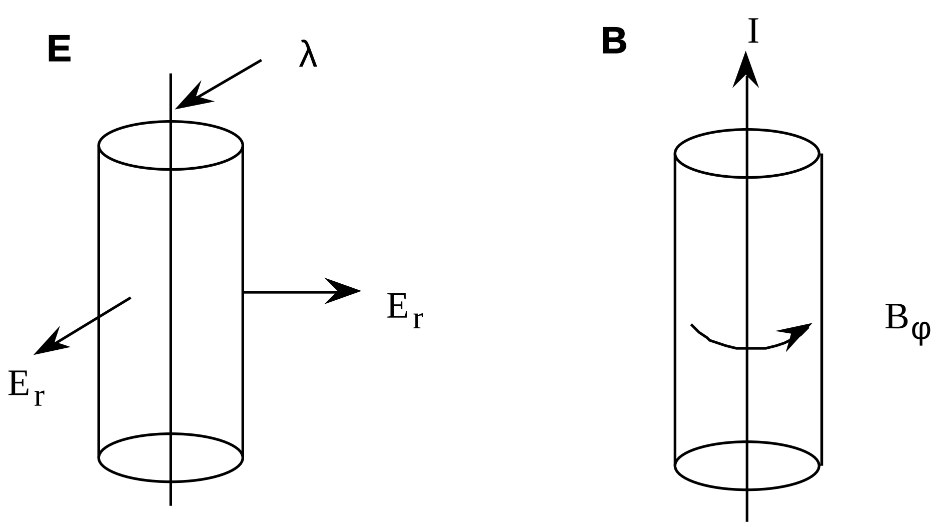
  E is irrotational $\nabla \times \mathbf{E} = 0$

  B is solenoidal $\nabla \cdot \mathbf{B} = 0$

- There are no magnetic monopoles → no point sources of **B**

  B produced by currents, currents produce only dipoles or higher multipoles.

Example: long straight wire



Absence of magnetic monopoles is experimental fact. If discovered, there would be point sources of **B**, and Maxwell's equations would have to be modified.

Searches have been made, in obscure places like bottom of ocean or moon dust. One event was reported B. Cabrera, Phys Rev Letters **48**, 1378 (1982), using a SQUID detector, but it was never reproduced.

Historical development is convoluted. Nowadays, define **B** by the force it produces on moving charge. This relates **B** to the early defining expts, which were on forces, not fields.

$$\mathbf{F} = \frac{q}{c}\left(\mathbf{v} \times \mathbf{B}\right) \qquad \text{Lorentz force, moving point charge}$$

$$d\mathbf{F} = \frac{I}{c}\, d\mathbf{l} \times \mathbf{B} \qquad \text{circuits}$$

$$\mathbf{F} = \frac{1}{c}\int_V \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\, d^3r \qquad \text{continuous distributions}$$

Actual measurements done only on closed circuits with forces:

$$\mathbf{F}_{12} = \frac{I_1\, I_2}{c^2} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \times \left(d\mathbf{l}_2 \times \left(\mathbf{r}_1 - \mathbf{r}_2\right)\right)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

Is it symmetric, $\mathbf{F}_{12} = \mathbf{F}_{21}$? Yes, vector identities

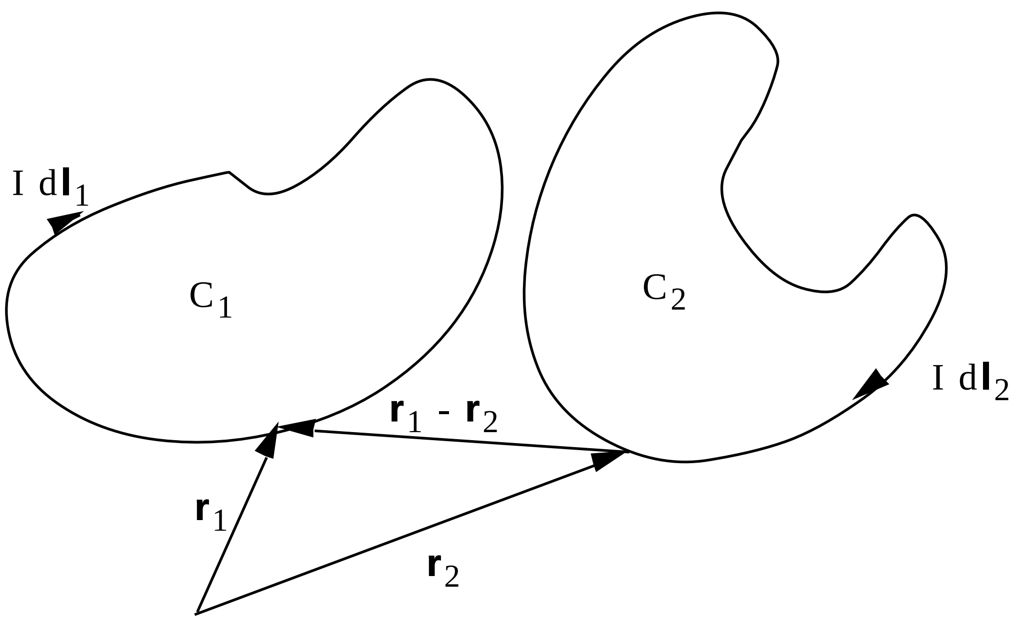
$$\mathbf{F}_{12} = -\frac{I_1 I_2}{c^2} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2 \left(\mathbf{r}_1 - \mathbf{r}_2\right)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

- Inverse square, as for Coulomb's law

- $\frac{1}{c}$ appears in all these expressions, c = speed of light. Some velocity is

needed, because **J** or **I** are proportional to velocity of charge, and this velocity must be removed to get the force. Meaning not known to early workers, simply a constant. But Maxwell realized that light could be described by propagating electromagnetic waves, c must be speed of light. First measurements of c were from magnetostatics, more accurate than direct speed measurements, because time intervals were too short to measure accurately then.

Geometry:



This form does not mention **B**. **B** is induced from $d\mathbf{F} = \frac{I}{c}\, d\mathbf{l} \times \mathbf{B}$:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c} \oint_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r' \quad \text{continuous distributions} \qquad \text{Biot-Savart law}$$

$$= \frac{I}{c} \oint_C \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad \text{circuits}$$

Then

$$\mathbf{F} = \frac{I_1}{c} \oint_{C_1} d\mathbf{l}_1 \times \frac{I_2}{c} \oint_{C_2} \frac{d\mathbf{l}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

- Same form as before, Biot-Savart law was correct form for **B** .

- Biot- Savart law and force laws are experimentally valid. Like Coulomb's law, they form the basis of all magnetostatic phenomena.

Note comparison of forms of **E** and **B** :

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c} \oint_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r' \qquad \text{Biot-Savart}$$

$$\mathbf{E}(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r' \qquad \text{Coulomb}$$

- The $\times$ gives Biot- Savart its solenoidal properties
- Both are inverse square
- Source of **E** is charge, source of **B** is current

Derive Maxwell's equations from Biot-Savart law:

- We distinguish between ∇ derivative wrt r

  and ∇' derivative wrt r´

Show $\nabla \cdot \mathbf{B} = 0$:

Product rule: $\nabla \times \left(f\mathbf{A}\right) = f\left(\nabla \times \mathbf{A}\right) - \mathbf{A} \times \nabla f$

$$\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right) = \frac{\nabla \times \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \mathbf{J}(\mathbf{r}') \times \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)$$

↑
0 because $\mathbf{J}(\mathbf{r}')$ not function of $\mathbf{r}$

$$\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -\frac{\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}$$

$$\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right) = \frac{\mathbf{J}(\mathbf{r}') \times \left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}$$

Substitute in Biot-Savart:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3r' = \frac{1}{c} \int_V \nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) d^3r'$$

$$= \nabla \times \left\{ \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r' \right\} = \nabla \times \mathbf{A}$$

Later, show this is $\mathbf{A}$ = vector potential

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad \text{div curl } \mathbf{A} = 0$$

- Equivalent to statement that there are no magnetic monopoles:

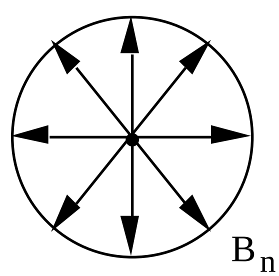
$$0 = \int_V \nabla \cdot \mathbf{B} \, d^3r = \int_S \mathbf{B} \cdot \mathbf{n} \, da = 4\pi \left(\text{magnetic charge enclosed}\right)$$

↑ div theorem ↑ Gauss law for magnetic fields

Recall $\mathbf{B} \sim \frac{1}{r^2}$

- equivalent to "net flux through <u>any</u> closed surface = 0"

- no point sources of $\mathbf{B}$



Show $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times \left\{ \nabla \times \left(\frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' \right) \right\} = \nabla \times (\nabla \times \mathbf{A})$$

$$= \nabla (\nabla \cdot \mathbf{A}) - \ {}^2\mathbf{A}$$

$$\nabla (\nabla \cdot \mathbf{A}) = \nabla \left(\nabla \cdot \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' \right)$$

$$= \nabla \left(\frac{1}{c} \oint_V \frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3r' \right)$$

↑ $0,\ \mathbf{J}(\mathbf{r}') \neq \mathbf{J}(\mathbf{r})$ ↑ $-\nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$

$$= -\frac{1}{c} \int_V \mathbf{J}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) d^3r' = \frac{1}{c} \oint_V \frac{\nabla' \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

↑ integrate by parts

$= 0$ $\quad \nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$ in steady state

$${}^2\mathbf{A} = {}^2 \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' = \frac{1}{c} \int_V \mathbf{J}(\mathbf{r}') \ {}^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

$$= \frac{1}{c} \int_V \mathbf{J}(\mathbf{r}') \left(- 4\pi \delta(\mathbf{r} - \mathbf{r}') \right) d^3r' = - \frac{4\pi}{c} \mathbf{J}(\mathbf{r})$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = - - \frac{4\pi}{c} \mathbf{J}(\mathbf{r}) = \frac{4\pi}{c} \mathbf{J}(\mathbf{r})$$ Maxwell equation for **B**

- Holds only in steady state, because we used $\nabla \cdot \mathbf{J} = 0$. Will have to be modified if $\mathbf{J} = \mathbf{J}(\mathbf{r}, t) \quad \mathbf{B} = \mathbf{B}(\mathbf{r}, t)$.
- Source of **B** is a current **J**.

Analogous to source of **E** is charge ρ $\quad \nabla \cdot \mathbf{E} = 4\pi \rho$

We now assume that all fields **B** are due to currents. What about permanent magnets? We will get to that.

Integral form of Maxwell's equation for **B** $\leftrightarrow$ Ampere's circuital law:

Stokes theorem or Curl theorem:

$$\int_S \left(\nabla \times \mathbf{V} \right) \cdot \mathbf{n} \, da = \int_C \mathbf{V} \cdot d\mathbf{l}$$

$$\int_S \left(\nabla \times \mathbf{B} \right) \cdot \mathbf{n} \, da = \int_C \mathbf{B} \cdot d\mathbf{l}$$

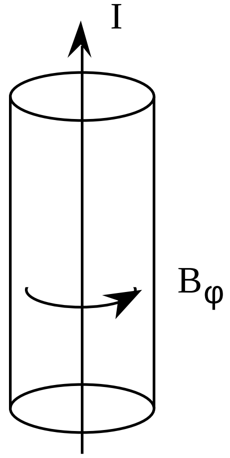
$$\oint_S \frac{4\pi}{c} \mathbf{J} \cdot \mathbf{n} \, da = \frac{4\pi}{c} I = \int_C \mathbf{B} \cdot d\mathbf{l}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c} I_{enc} = \frac{4\pi}{c} \int_S \mathbf{J} \cdot \mathbf{n}\, da$$ $\left\{ \begin{array}{l} \text{Ampere's law} \\ \text{Integral form of Maxwell's Eq} \end{array} \right.$

Use for cases of high symmetry, as for Gauss law.

Examples:

- long straight wire



$$\oint_C \mathbf{B} \cdot d\mathbf{l} = 2\pi\, r\, B_\varphi = \frac{4\pi}{c} I$$

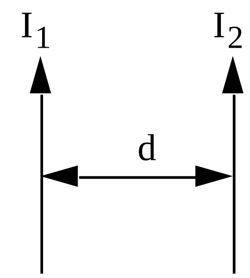
$$B_\varphi = \frac{2I}{c} \frac{1}{r}$$

In vector notation, $\mathbf{B} = \dfrac{2}{c} \dfrac{\mathbf{I} \times \mathbf{r}}{r^2}$

- cylinder of radius a, current density J, when $r < a$,

$$B_\varphi = \frac{2\pi}{c} J\, r \qquad \rightarrow \qquad \mathbf{B} = \frac{2\pi}{c} \mathbf{J} \times \mathbf{r}$$

- force between 2 wires separated by d:



$$B_\varphi(1) = \frac{2I_2}{c}\ \frac{1}{d}$$

$$d\mathbf{F}(1) = \frac{I_1}{c}\, d\mathbf{l} \times \mathbf{B}(1) \;=\; \frac{I_1}{c}\, dl\ \frac{2I_2}{c}\ \frac{1}{d}$$

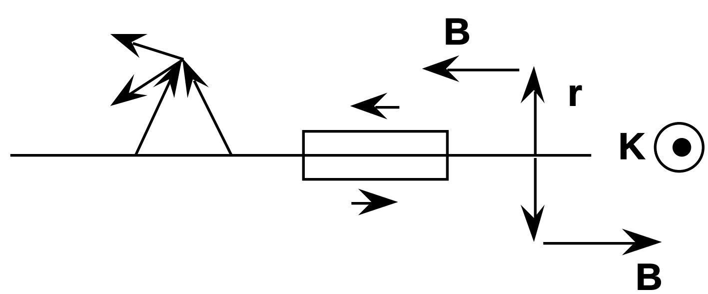
$$\frac{F}{l} = \frac{2\, I_1\, I_2}{c^2}\ \frac{1}{d}$$

This is what Biot and Savart measured.

- Surface current **K** Definition:

if I flows in cross section A, then $J = \frac{I}{A}$ = current/unit area I

if I flows in surface layer, then $K = \frac{I}{L}$ = current/unit length **I**



B must be || surface, all other components are canceled by symmetry.

Draw Amperian contour $\int_C \mathbf{B} \cdot d\mathbf{l} = B\,L + B\,L = \frac{4\pi}{c} K\,L$

$\mathbf{B} = \frac{2\pi}{c} \mathbf{K} \times \mathbf{n}$ $\quad$ $\mathbf{n}$ is unit normal in direction of observation point.

- Ends do not contribute by symmetry, so thickness of box does not matter

  $\mathbf{B}$ does not depend on distance from surface

Infinite solenoid:

N turns/unit length, each carrying current I, Amperian contour of length L

$$\int_C \mathbf{B} \cdot d\mathbf{l} = B_{in}\,L + B_{out}\,L$$

- infinite length $\qquad$ B_{in} cannot vary along length

  $\nabla \cdot \mathbf{B} = 0 = \nabla \times \mathbf{B}$ inside $\qquad$ B_{in} cannot vary with radius (homework)

- infinite length $\quad B_{out} = 0 \quad$ Not intuitive, as there should be return field

in any finite solenoid. Lines of field cannot end, but they return at infinity here. Can be proven with difficult integrals. Accept this fact, then

$$B_{in}\,L = \frac{4\pi}{c}\text{ current enclosed} = \frac{4\pi}{c} N\,I\,L$$

$$B_{in} = \frac{4\pi}{c} N\,I$$

Griffiths gives many more examples of Ampere's law.

We have Maxwell's eq for $\mathbf{B}$:

$\nabla \cdot \mathbf{B} = 0$ $\qquad$ no magnetic monopoles

$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}$ $\qquad$ Ampere's law

Potentials for **B**

In analogy with **E**, define a potential from which **B** can be derived.

Two cases:

1. **J** = 0 then $\nabla \times \mathbf{B} = 0$ and $\mathbf{B} = -\nabla\varphi_M$.

   All metods of electrostatics go over to magnetostatics.

   This is often easiest method, because the vector character is simple.

2. **J** ≠ 0 Then use $\nabla \cdot \mathbf{B} = 0$.

   Vector identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ arbitrary **A**

   Expect $\mathbf{B} = \nabla \times \mathbf{A}$

Previously showed: starting with Biot-Savart,

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r'$$

product rule: $\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right) = \underset{\substack{\uparrow \\ 0}}{\frac{\nabla \times \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}} - \mathbf{J}(\mathbf{r}') \times \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)$

0 because **J**(**r**´) not function of **r**

$$\nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = - \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad \nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'$$

Define $$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'$$

Compare $$\varphi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'$$

- vector $\mathbf{J}$, scalar ρ
- $\mathbf{B} = \nabla \times \mathbf{A}$ "solenoidal" $\quad$ $\mathbf{E} = - \nabla\varphi$ "irrotational"

Alternate derivation, valid only in Cartesian coordinates;

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} \qquad \mathbf{B} = \nabla \times \mathbf{A} \qquad \nabla \times (\nabla \times \mathbf{A}) = \frac{4\pi}{c} \mathbf{J}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

We will show next that $\nabla \cdot \mathbf{A} = 0$ if we wish.

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J}$$

Recall for electrostatics

$$\nabla^2 \varphi = -4\pi\rho$$

$$\varphi = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

Same scalar equation for each component A_i

$$\nabla^2 A_i = -\frac{4\pi}{c} J_i$$

By analogy, solution for A_i is

$$A_i(\mathbf{r}) = \frac{1}{c}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

• Derivation holds only in Cartesian coordinates, because we had to take components. Only in Cartesian are the unit vector directions independent of coordinates.

With this expression for $\mathbf{A}$, $\nabla \cdot \mathbf{A} = \nabla \cdot \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$

$$= \frac{1}{c}\int_V \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right) d^3r' = \frac{1}{c}\int_V \left[\frac{\nabla \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \mathbf{J}(\mathbf{r}') \cdot \nabla \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] d^3r'$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$0,\ \mathbf{J}\neq\mathbf{J}(\mathbf{r}) \qquad\qquad -\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$$

$$= \frac{1}{c}\int_{V} \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' \ = \ 0$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

integrate by parts steady currents, $\nabla\cdot\mathbf{J} = 0$

Gauge transformations

Since $\mathbf{B} = \nabla\times\mathbf{A}$, any vector function with zero curl can be added to $\mathbf{A}$ without changing $\mathbf{B}$. Gradients have zero curl, so $\nabla\psi$ can be added to $\mathbf{A}$ without changing $\mathbf{B}$.

• Analgous to adding const to φ without changing $\mathbf{E}$. This is "trivial" addition because constant has no spatial derivatives, so spatial variation of φ is not changed.

However, $\nabla\psi$ has spatial variation, and could change spatial variation of $\mathbf{A}$, in particular $\nabla\cdot\mathbf{A}$. By adding $\nabla\psi$, we can choose value of $\nabla\cdot\mathbf{A}$, called the gauge of $\mathbf{A}$.

• $\nabla\cdot\mathbf{A}$ is the conventional way to specify the gauge, because if $\nabla\cdot\mathbf{A}$ is known, then both the div and curl of $\mathbf{A}$ are specified $\mathbf{A}$ is uniquely specified, provided $\mathbf{A}$ goes to zero at ∞ (Helmholtz theorem).

• Choice $\nabla\cdot\mathbf{A} = 0$ is called Coulomb gauge.

$\nabla\cdot\mathbf{A}$ will be preserved, i. e., the gauge will be preserved, if we restrict ψ to scalar functions satisfying Laplace's equation:

$$\nabla\cdot\mathbf{A}' = \nabla\cdot\left(\mathbf{A}+\nabla\psi\right) = \nabla\cdot\mathbf{A} + \nabla\cdot\left(\nabla\psi\right) = \nabla\cdot\mathbf{A} + {}^2\psi$$

Thus, $\nabla \cdot \mathbf{A}$, whatever its value, is preserved if ${}^{2}\psi = 0$

Compare electrostatics:

$$\varphi = \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' \ + \ \text{const}$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' \ + \nabla\psi$$

• $\nabla\psi$ allows more flexibility on $\mathbf{A}$ than const does for φ.

Maxwell's equations for $\mathbf{A}$:

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$

$$\nabla\left(\nabla \cdot \mathbf{A}\right) - {}^{2}\mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

$${}^{2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{J} \qquad \text{in Coulomb gauge, } \nabla \cdot \mathbf{A} = 0$$

$\nabla \cdot \mathbf{B} = 0$ is automatically satisfied: $\mathbf{B} = \nabla \times \mathbf{A}$ and $\nabla \cdot \left(\nabla \times \mathbf{A}\right) = 0$

• Similarity to electrostatics

$${}^{2}\varphi = -4\pi\rho$$

$${}^{2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}$$

Laplacian of potential gives source, with appropriate constant. For magnetostatics, constant has dimensioins of velocity to remove velocity in source. **A** is not time dependent in magnetostatics (it will be in electrodynamics).

• In classical physics, we usually say that the potentials are alternate descriptions of the fields and contain no more information than **E** or **B**. Normally, the fields are takent to be the more physically relevant quantities. However, in quantum mechanics, the potenial A is not simply an alternative, it is required to generalize the Schroedinger equation for magnetic fields.

$$H\psi = p^2/2m = \frac{\left(\frac{h}{2\pi i}\nabla\right)^2}{2m} = E\psi$$

$$H\psi = \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2/2m = \frac{\left(\frac{h}{2\pi i}\nabla - \frac{e}{c}\mathbf{A}\right)^2}{2m} = E\psi$$

$\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)$ is the momentum canonical to **r** if magnetic fields are present. The canonical momentum is determined by writing the Hamiltonian H with the magnetic field and using the definition $\mathbf{p} = \frac{\partial H}{\partial \mathbf{r}}$.

For multiply connected geometries, is is possible to create electron trajectories that do not encounter any magnetic field, but which enclose a magnetic field with finite flux. It turns out that the interference of two electrons whose trajectories see not field but whose paths pass on opposite sides of a finite field region is strongly affected by the enclosed field. Aharanov-Bohm effect, proven conclusively in experiments by Tonomura using superconductors to shield the fields, in 1980's.

Multipole expansion of **A**

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

Taylor series expansion $\dfrac{1}{|\mathbf{r} - \mathbf{r}'|} = \dfrac{1}{|\mathbf{r}|} + \dfrac{\mathbf{r} \cdot \mathbf{r}'}{|\mathbf{r}|^3} + \ldots$

Exactly as in electrostatics

$$\mathbf{A}(\mathbf{r}) = \underbrace{\frac{1}{c}\,\frac{\int_V \mathbf{J}(\mathbf{r}')\, d^3r'}{r}}_{\text{monopole}} + \underbrace{\frac{1}{c}\,\frac{\int_V \mathbf{J}(\mathbf{r}')\, \mathbf{r}' \cdot \mathbf{r}\, d^3r'}{r^3}}_{\text{dipole}} + \underbrace{\ldots}_{\text{quadrupole}}$$

Monopole is zero for steady currents

- Physical reason: currents must circulate in closed paths, average value of $\mathbf{J}$ is zero
- Formal proof:

$$\nabla' \cdot \left(x'_i\, \mathbf{J}(\mathbf{r}')\right) = \underbrace{x'_i \nabla' \cdot \mathbf{J}(\mathbf{r}')}_{0,\ \text{steady currents}} + \underbrace{\mathbf{J}(\mathbf{r}') \cdot \nabla' x'_i}_{J_i(\mathbf{r}')}$$

$$\int_V J_i(\mathbf{r}')\, d^3r' = \int_V \nabla' \cdot \left(x'_i\, \mathbf{J}(\mathbf{r}')\right) d^3r' = \int_S \mathbf{n} \cdot \left(x'_i\, \mathbf{J}(\mathbf{r}')\right) da'$$

For a large enough surface, containing all the currents, $\mathbf{J} = 0$

QED

Thus, no monopole term in $\mathbf{A}$.

• Expected by analogy to electrostatics, since monopole term is total charge, and there are no magnetic charges.

Dipole term:

Want to define a dipole moment for $\mathbf{J}$. But simplest form, for electrostatics, won't work. We had $\mathbf{p} = \int_V \mathbf{r}'\, \rho(\mathbf{r}')\, d^3r'$, and dipole term in potential was $\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} = \frac{\int_V \rho(\mathbf{r}')\, \mathbf{r}' \cdot \mathbf{r}\, d^3r'}{r^3}$ similar to above. But, we cannot pull out the $\cdot\,\mathbf{r}$ term, becuase it would leave a tensor $\mathbf{J}(\mathbf{r}')\mathbf{r}'$. Therefore we must get a vector out of $\mathbf{J}$ and $\mathbf{r}'$, turns out to be $\mathbf{r}' \times \mathbf{J}(\mathbf{r}')$. Vectorology required to transform the dipole to this form relies on the trick we used above:

$$\int_V J_i(\mathbf{r}') \left(\mathbf{r} \cdot \mathbf{r}'\right) d^3r' = \int_V \left(\nabla' \cdot\; x'_i\, \mathbf{J}(\mathbf{r}')\right) \left(\mathbf{r} \cdot \mathbf{r}'\right) d^3r'$$

$$= - \int_V x'_i\, \mathbf{J}(\mathbf{r}') \cdot \nabla'\left(\mathbf{r} \cdot \mathbf{r}'\right) d^3r'$$

$$= - \int_V x'_i\, \mathbf{J}(\mathbf{r}') \cdot \mathbf{r}\, d^3r'$$

In vector notation,

$$\int_V \mathbf{J}(\mathbf{r}') \left(\mathbf{r} \cdot \mathbf{r}'\right) d^3r' = - \int_V \mathbf{r}' \left(\mathbf{J}(\mathbf{r}') \cdot \mathbf{r}\right) d^3r'$$

• Only valid for steady currents, since we used $\nabla \cdot \mathbf{J} = 0$ to get the first step.

$$\text{Now} \int_V \mathbf{r} \times \left(\mathbf{r}' \times \mathbf{J}(\mathbf{r}')\right) d^3r' = \int_V \mathbf{r}'\left(\mathbf{r} \cdot \mathbf{J}(\mathbf{r}')\right) - \mathbf{J}(\mathbf{r}') \left(\mathbf{r} \cdot \mathbf{r}'\right) d^3r' \qquad \text{(BAC - CAB)}$$

$$= -2 \int_V \mathbf{J}(\mathbf{r}') \left(\mathbf{r} \cdot \mathbf{r}'\right) d^3r'$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \frac{\int_V \mathbf{J}(\mathbf{r}')\, \mathbf{r}' \cdot \mathbf{r}\, d^3r'}{r^3} = \frac{-\frac{1}{2c} \int_V \mathbf{r} \times \left(\mathbf{r}' \times \mathbf{J}(\mathbf{r}')\right) d^3r'}{r^3}$$

define $\quad \mathbf{m} = \dfrac{1}{2c} \displaystyle\int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, d^3r' \quad$ magnetic dipole moment

Then $\quad \mathbf{A}(\mathbf{r}) = \underset{\substack{\uparrow \\ \text{monopole}}}{0} + \underset{\substack{\uparrow \\ \text{dipole}}}{\dfrac{\mathbf{m} \times \mathbf{r}}{r^3}} + \underset{\substack{\uparrow \\ \text{quadrupole}}}{\dots}$

- Similarities to electric dipole:

$$\mathbf{m} = \frac{1}{2c} \int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, d^3r'$$

$$\mathbf{p} = \int_V \mathbf{r}'\, \rho(\mathbf{r}')\, d^3r'$$

- same form: $\mathbf{r}'$ times source
  - simple product for $\mathbf{p}$ replaced by cross product for $\mathbf{m}$
  - $\mathbf{m}$ has $\dfrac{1}{c}$ prefactor to normalize out the velocity in $\mathbf{J}$
  - $\mathbf{m}$ has $\dfrac{1}{2}$ prefactor

$$\varphi(\mathbf{r}) = \underset{\substack{\uparrow \\ \text{monopole}}}{\frac{Q}{r}} + \underset{\substack{\uparrow \\ \text{dipole}}}{\frac{\mathbf{p} \cdot \mathbf{r}}{r^3}} + \underset{\substack{\uparrow \\ \text{quadrupole}}}{\dots}$$

$$\mathbf{A}(\mathbf{r}) = \underset{\downarrow}{0} + \underset{\downarrow}{\frac{\mathbf{m}\times\mathbf{r}}{r^3}} + \underset{\downarrow}{\dots}$$

- "dots" are replaced by "crosses"
- monopole missing from magnetic case

Amazing feature: in spite of differences in form, and $\nabla\varphi$ vs $\nabla\times\mathbf{A}$, the fields produced by electric and magnetic dipoles are the same.

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right) = \nabla\times\left\{\left(\frac{1}{r^3}\right)\mathbf{m}\times\mathbf{r}\right\} \underset{\uparrow}{=} \underbrace{\frac{\nabla\times(\mathbf{m}\times\mathbf{r})}{r^3}}_{①} - \underbrace{(\mathbf{m}\times\mathbf{r})\times\nabla\left(\frac{1}{r^3}\right)}_{②}$$

product rule $\nabla\times(f\,\mathbf{A}) = f\,\nabla\times\mathbf{A} - \mathbf{A}\times\nabla f$

① : product rule : $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

$$\nabla\times(\mathbf{m}\times\mathbf{r}) = \underset{\substack{\uparrow\\0}}{(\mathbf{r}\cdot\nabla)\,\mathbf{m}} - \underset{\substack{\uparrow\\\mathbf{m}}}{(\mathbf{m}\cdot\nabla)\,\mathbf{r}} + \underset{\substack{\uparrow\\3}}{\mathbf{m}\,(\nabla\cdot\mathbf{r})} - \underset{\substack{\uparrow\\0}}{\mathbf{r}\,(\nabla\cdot\mathbf{m})}$$

$\mathbf{m}$ is not a function of $\mathbf{r}$ $\uparrow$

$$(\mathbf{m}\cdot\nabla)\,\mathbf{r} = \left(m_x\frac{\partial}{\partial x} + m_y\frac{\partial}{\partial y} + m_z\frac{\partial}{\partial z}\right)\mathbf{r} = m_x\,\mathbf{i} + m_y\,\mathbf{j} + m_z\,\mathbf{k} = \mathbf{m}$$

$$① = \frac{2\,\mathbf{m}}{r^3}$$

② : $$\nabla\left(\frac{1}{r^3}\right) = -3\,\frac{1}{r^4}\,\hat{\mathbf{r}} = -3\,\frac{\mathbf{r}}{r^5}$$

$$(\mathbf{m}\times\mathbf{r})\times\mathbf{r} = -\,\mathbf{r}\times(\mathbf{m}\times\mathbf{r}) = -\,\mathbf{m}\,(\mathbf{r}\cdot\mathbf{r}) + \mathbf{r}\,(\mathbf{m}\cdot\mathbf{r}) \qquad \text{BAC - CAB}$$

$$= -\,\mathbf{m}\,r^2 + \mathbf{r}\,(\mathbf{m}\cdot\mathbf{r})$$

$$\left(\mathbf{m} \times \mathbf{r}\right) \times \nabla\left(\frac{1}{r^3}\right) = 3\ \frac{\mathbf{m}}{r^3} - \frac{3\left(\mathbf{m} \cdot \mathbf{r}\right)\mathbf{r}}{r^5}$$

$$\mathbf{B} = ① - ② = \frac{3\left(\mathbf{m} \cdot \mathbf{r}\right)\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}$$

- same form as for electric dipole

$$\mathbf{E} = \frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}$$

Interpretation of **m** for flat circuit:

$$\mathbf{m} = \frac{1}{2c} \int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\ d^3r' \ \rightarrow\ \frac{1}{2\ c}\ I \int_C \mathbf{r} \times d\mathbf{l}$$

$$\frac{\mathbf{r} \times d\mathbf{l}}{2} = \frac{1}{2}\left(\underset{\substack{\uparrow \\ \text{height}}}{r\ \sin\theta}\right)\underset{\substack{\uparrow \\ \text{base}}}{dl}$$

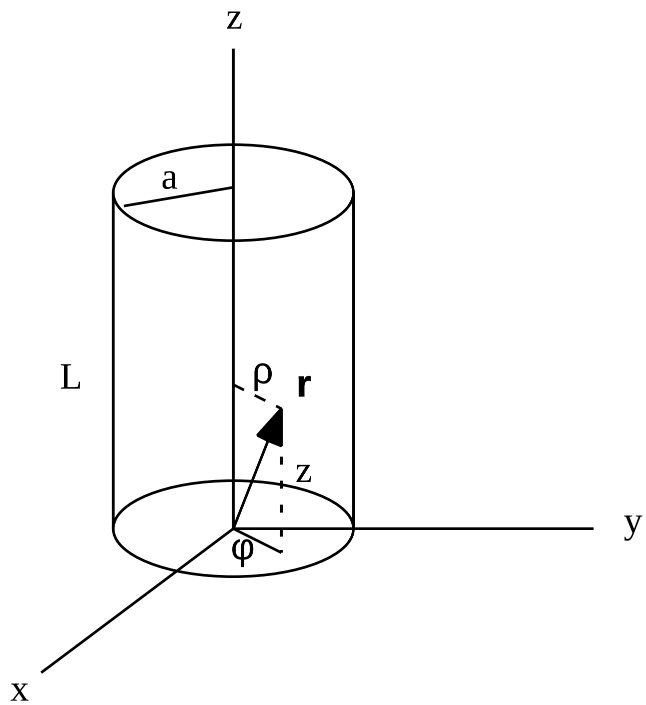
$$= da$$

$m = \dfrac{I}{c}$ area   direction of **m** given by right hand rule applied to I

Examples of dipole moments:

Surface charge on rotating cylinder of radius a and length L

use cylindrical coordinates



$\rho(\mathbf{r}) = \sigma\, \delta(\rho - a)$ $\quad \mathbf{v}(\mathbf{r}) = \omega\, \rho\, \varphi$ $\quad \mathbf{J}(\mathbf{r}) = \rho\, \mathbf{v} = \sigma\, \delta(\rho - a)\, \omega\, \rho\, \phi = \sigma\, \omega\, \rho\, \delta(\rho - a)\, \phi$

$$\mathbf{m} = \frac{1}{2c} \int_V \mathbf{r} \times \mathbf{J}(\mathbf{r})\, d^3r = \frac{1}{2c} \int_V \left(\rho\, \hat{\rho} + z\, \mathbf{k}\right) \times \sigma\, \omega\, \rho\, \delta(\rho - a)\, \phi\, d^3r$$

$$= \frac{1}{2c} \int_V \sigma\, \omega\, \rho^2\, \delta(\rho - a)\, \hat{\rho} \times \phi\, d^3r + \frac{1}{2c} \int_V \sigma\, \omega\, \rho\, z\, \delta(\rho - a)\, \mathbf{k} \times \phi\, d^3r$$

↑ $\qquad\qquad\qquad\qquad$ ↑

$\mathbf{k}$ $\qquad\qquad\qquad\qquad$ $-\hat{\rho} = -\cos\varphi\, \mathbf{i} + \sin\varphi\, \mathbf{j}$

$$= \mathbf{k}\, \frac{\sigma\, \omega}{2c} \int_V \rho^2\, \delta(\rho - a)\, d\rho\, \rho d\varphi\, dz - \frac{\sigma\, \omega\, z}{2c} \int_V \rho\, \delta(\rho - a) \left(\cos\varphi\, \mathbf{i} + \sin\varphi\, \mathbf{j}\right) d\rho\, \rho d\varphi\, dz$$

↑

$= \mathbf{k}\, \dfrac{\sigma\, \omega}{2c}\, 2\pi L\, a^3$ $\qquad\qquad$ 0 by integration over $d\varphi$

$$= \mathbf{k}\, \frac{\sigma\, \omega\, a\, L}{c}\, \pi a^2$$

Note this is the form for a flat circuit: $\sigma\,\omega\,a = \frac{\sigma\, 2\pi\, a}{T}$ = surface current/length

$\mathbf{m} = \mathbf{k}\,\frac{I}{c}\,A = \mathbf{k}\,\frac{\sigma\,\omega\,a}{c}\,\pi a^2$ /unit length $\qquad$ total $\mathbf{m} = \mathbf{k}\,\frac{\sigma\,\omega\,a\,L}{c}\,\pi a^2$

Other variations: uniform volume charge density, and surface and volume charge densities on sphere.

We will not derive energy of magnetic fields in magnetostatics, because setting

up fields means setting up currents, which involves $\frac{\partial \mathbf{J}}{\partial t}$. Defer to chapter on

time

varying fields. Result is

$$W = -\,\mathbf{m}\cdot\mathbf{B} \qquad \text{where } \mathbf{B} \text{ is an external field.}$$

- Same form as for electric dipole. Then

$$\mathbf{F} = -\,\nabla\, W = \nabla\left(\mathbf{m}\cdot\mathbf{B}\right)$$

$$\tau = -\,\frac{\partial W}{\partial\theta} = -\,\frac{\partial}{\partial\theta}\, m\,B\cos\theta = m\,B\sin\theta = |\,\mathbf{m}\times\mathbf{B}\,|$$

Connection between angular momentum and magnetic moment:

$$\mathbf{J} = n\,q\,\mathbf{v} \qquad\qquad \mathbf{p} = m\mathbf{v} \qquad \mathbf{J} = \frac{n\,q}{m}\,\mathbf{p}$$

$$\mathbf{m} = \frac{1}{2c}\int_V \mathbf{r} \times \mathbf{J}(\mathbf{r})\, d^3r = \frac{1}{2c}\int_V \mathbf{r} \times \frac{n\,q}{m}\mathbf{p}\, d^3r = \frac{q}{2mc}\int_V \mathbf{r} \times \underset{\substack{\uparrow \\ \text{density of momentum}}}{n\mathbf{p}}\, d^3r$$

$$= \frac{q}{2mc}\mathbf{L}$$

In QM, same relation holds. L_z , L^2 are quantized. Smallest unit of L_z is $\frac{h}{2\pi}$.

Thus, smallest unit of m_z is $\frac{q}{2mc}\frac{h}{2\pi} = \frac{e\frac{h}{2\pi}}{2mc} = \beta$ = Bohr magneton.

Above holds for orbital angular momentum. There is also spin angular momentum, which, for electrons (spin 1/2), has one Bohr magneton associated with it. However origin of spin moment is not an orbiting charge, but a relativistic quantum effect. Nevertheless, it obeys the quantization rule that the smallest unit of magnetic moment is the Bohr magneton.

Effect of magnetized matter (from Jackson)

We have just seen, in the multipole expansion, that a point dipole produces fields and potentials at distant points. However, in describing **B**, we assumed the only source was a current. Apparently a distribution of point dipoles (as in magentized matter) will also produce **B**. To describe the **B** so produced, proceed in analogy to φ from ρ, or **A** from **J**.

dipole at origin: $\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m} \times \mathbf{r}}{r^3}$

dipole at $\mathbf{r}'$: $\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$

$\mathbf{M}(\mathbf{r})$ = density of magnetic dipole moment at $\mathbf{r}$

$\mathbf{M}(\mathbf{r})\, d^3r$ = total dipole moment in volume element d^3r

$= \sum_i \mathbf{m}_i$ all i in d^3r

$\mathbf{M}(\mathbf{r})$ is the variation of dipole moment density in space

$\mathbf{A}(\mathbf{r})$ due to the density $\mathbf{M}(\mathbf{r})$ is

$$\mathbf{A}(\mathbf{r}) = \int_V \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r'$$

This is equivalent to a "magnetizaion current" :

$$\mathbf{A}(\mathbf{r}) = \int_V \mathbf{M}(\mathbf{r}') \times \nabla'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) d^3r' \qquad \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)$$

Integrate by parts: product rule

$$\int_V \nabla' \times \left(\frac{\mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}\right) d^3r' = \int_V \frac{\nabla' \times \mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' - \int_V \mathbf{M} \times \nabla'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) d^3r'$$

Corollary to Divergece theorem:

$$\int_S \mathbf{n}' \times \left(\frac{\mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}\right) da' = \int_V \frac{\nabla' \times \mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' - \int_V \mathbf{M} \times \nabla'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) d^3r'$$

Rearange:

$$\mathbf{A}(\mathbf{r}) = \int_V \mathbf{M}(\mathbf{r}') \times \nabla'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) d^3r' = \int_V \frac{\nabla' \times \mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' - \int_S \mathbf{n}' \times \left(\frac{\mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}\right) da'$$

$$= \int_V \frac{\nabla' \times \mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' + \int_S \frac{\mathbf{M} \times \mathbf{n}'}{|\mathbf{r} - \mathbf{r}'|}\, da'$$

Potential is the same as if $\mathbf{M}(\mathbf{r})$ were replaced by a volume and surface current:

$$\mathbf{J}_M(\mathbf{r}) = c\nabla \times \mathbf{M}$$

$$\mathbf{K}_M(\mathbf{r}) = c\mathbf{M} \times \mathbf{n}$$

Then

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}_M(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' + \frac{1}{c}\int_S \frac{\mathbf{K}_M(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, da'$$

Thus every distribution $\mathbf{M}(\mathbf{r})$ is equivalent to the currents

$$\mathbf{J}_M(\mathbf{r}) = c\, \nabla \times \mathbf{M}(\mathbf{r})$$
$$\mathbf{K}_M(\mathbf{r}) = c\mathbf{M}(\mathbf{r}) \times \mathbf{n}$$

ie, the equivalent currents produce the same $\mathbf{B}$ and $\mathbf{A}$ as the distribution $\mathbf{M}(\mathbf{r})$.

If there are both free currents $\mathbf{J}_{free}$ and magnetization currents $\mathbf{J}_M$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r' + \frac{1}{c}\int_V \frac{\mathbf{J}_M(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

Each source satisfies Maxwell equations

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}_{free} + \frac{4\pi}{c}\mathbf{J}_M = \frac{4\pi}{c}\mathbf{J}_{free} + \frac{4\pi}{c} c\, \nabla \times \mathbf{M}$$

$$= \frac{4\pi}{c}\mathbf{J}_{free} + 4\pi \nabla \times \mathbf{M}$$

$$\nabla \times \left(\mathbf{B} - 4\pi \mathbf{M}\right) = \frac{4\pi}{c}\mathbf{J}_{free}$$

Define $\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{free}$

- $\mathbf{H}$ is the field of the external sources, $\mathbf{B}$ is the field of the sources + magnets
- $\mathbf{H}$ is the field we would have gotten without the magnetic material
- $\nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{B} - 4\pi \nabla \times \mathbf{M} = - 4\pi \nabla \times \mathbf{M} \neq 0$ especially at boundaries where $\mathbf{M}$ has step function

This solves problem for permanently magnetized material. Given $\mathbf{J}_{free}$ and $\mathbf{M}(\mathbf{r})$, Maxwells equations $\quad \nabla \cdot \mathbf{B} = 0$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\,\mathbf{J}_{free} + 4\pi\,\nabla \times \mathbf{M}$$

or the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\oint_V \frac{\mathbf{J}_{free}(\mathbf{r}') \times \left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r' + \frac{1}{c}\oint_V \frac{\left(c\ \nabla \times \mathbf{M}\right) \times \left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r'$$

$\mathbf{M}(\mathbf{r})$ acts as a source, and belongs naturally on RHS.

Problems will demonstrate how to solve for the fields when a permanent dipole distribution is present. Use equivalent currents.

What about induced magnetism? Assume $\mathbf{M} = \chi\,\mathbf{H}$, analogy to dielectrics.

Then $\mathbf{B} = \mathbf{H} + 4\pi\,\mathbf{M} = \left(1 + 4\pi\chi\right)\mathbf{H} = \mu\,\mathbf{H}$

Then $\nabla \times \mathbf{H} = \frac{4\pi}{c}\,\mathbf{J}_{free}$ or $\mathbf{H}(\mathbf{r}) = \frac{1}{c}\oint_V \frac{\mathbf{J}_{free}(\mathbf{r}') \times \left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3r'$

solves for $\mathbf{H}$ and $\mathbf{M} = \chi\,\mathbf{H}$, $\mathbf{B} = \mu\,\mathbf{H}$ solves for $\mathbf{M}$ and $\mathbf{B}$.

In this case, $\mathbf{M}$ is a response to the applied source field, and belongs naturally on LHS.

Example of uniformly magnetized cylinder:

$\mathbf{M}(\mathbf{r}) = \mathbf{M} = M\,\mathbf{k}$ for $\mathbf{r}$ inside a cylinder of radius a, length 2 L. Then

$\mathbf{J}_M = c\nabla \times \mathbf{M} = 0$

$\mathbf{K}_M = c\ \mathbf{M} \times \mathbf{n} = c\ M\ \hat{\phi}$

Uniformly magnetized cylinder is equivalent to a solenoid.

Analogy to **E**

$$\nabla \cdot \mathbf{D} = 4\pi \rho_{free} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = 0 \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{free}$$

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \qquad \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$$

linear media:

$$\mathbf{P} = \chi \mathbf{E} \qquad \mathbf{M} = \chi \mathbf{H}$$

$$\mathbf{D} = \kappa \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

Boundary conditions

$$\nabla \cdot \mathbf{B} = 0 \qquad \int_S \mathbf{B} \cdot \mathbf{n}\, da = 0 \qquad \left(B_{n2} - B_{n1}\right) A = 0 \qquad B_{n2} = B_{n1}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} \qquad \int_C \mathbf{H} \cdot d\mathbf{l} = \frac{4\pi}{c} \int_S \mathbf{J} \cdot \mathbf{n}\, da$$

$\left(H_{t2} - H_{t1}\right) L = \frac{4\pi}{c} K_{free} L$ $\qquad$ K_{free} = surface current density of free charges

To keep directions straight, define **n** as pointing into 2. Then

$$\mathbf{n} \times \left(\mathbf{H}_2 - \mathbf{H}_1\right) = \frac{4\pi}{c} \mathbf{K}$$

Examples:

Infinitely long uniformly magnetized rod

Equivalent current densities: $\mathbf{J}_M = c\, \nabla \times \mathbf{M} = 0$

$$\mathbf{K}_M = c\, \mathbf{M} \times \mathbf{n} = c\, M\, \hat{\phi}$$

Same as infinitely long solenoid. Find B with Ampere's law on equivalent problem:

$$\int_C \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c} \int_S \mathbf{J} \cdot \mathbf{n}\, da$$

$$\underset{\text{inside}}{\underset{\uparrow}{B\,L}} + \underset{\text{outside}}{\underset{\uparrow}{0}} = \frac{4\pi}{c} K\,L = \frac{4\pi}{c} c\,M\,L = 4\pi\,M\,L$$

$\mathbf{B} = 4\pi\,\mathbf{M}$ inside

$\mathbf{H} = \mathbf{B} - 4\pi\,\mathbf{M} = 0$ inside

$\mathbf{B} = \mathbf{H} = \mathbf{M} = 0$ outside

No return field because the bar is infinitely long.

Plane boundary between μ_1 and μ_2

in 1, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{M}$ are known, $\mathbf{B} = \mu_1 \mathbf{H}$ $\quad \mathbf{M} = \dfrac{\mathbf{B} - \mathbf{H}}{4\pi}$

What are $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ in 2?

2 BC give 2 components of $\mathbf{B}$ (or $\mathbf{H}$)

other vectors given by $\mathbf{B} = \mu_2 \mathbf{H}$, $\mathbf{M} = \dfrac{\mathbf{B} - \mathbf{H}}{4\pi}$

Assume no free surface currents.

$B_{n2} = B_{n1}$

$B_{t2} = \mu_2\, H_{t2} = \mu_2\, H_{t1} = \mu_2 \dfrac{B_{t1}}{\mu_1}$

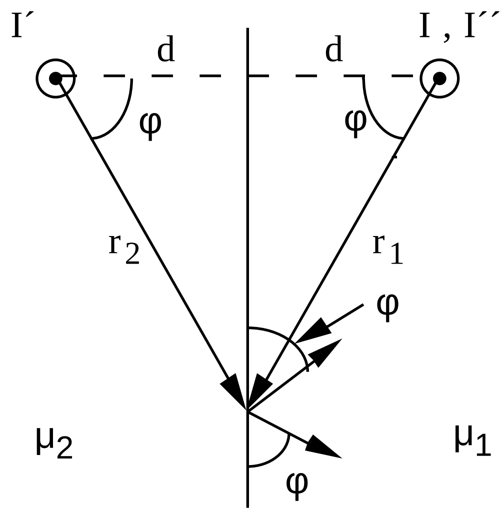
This determines $\mathbf{B}_2$ in terms of $\mathbf{B}_1$. $\mathbf{H}_2 = \dfrac{\mathbf{B}_2}{\mu_2}$ $\quad \mathbf{M}_2 = \dfrac{\mathbf{B}_2 - \mathbf{H}_2}{4\pi}$

Note if α = angle of $\mathbf{B}$ with respect to boundary plane,

$$\tan\alpha_2 = \frac{B_{n2}}{B_{t2}} = \frac{\mu_1}{\mu_2}\frac{B_{n1}}{B_{t1}} = \frac{\mu_1}{\mu_2}\tan\alpha_1$$

long straight wire in a permeable medium:

Straight wire near a boundary



Use image currents to satisfy boundary conditions

In 1, field is due to I and its image I´

In 2, field is due to I´´

Use Ampere's law to find **H**, the field of the free currents, use **B** = μ**H** to find **B.**

Adjust I´and I´´ to satisfy BC

On boundary, $r_1 = r_2 = r$

$$H_{t1} = -\frac{2I}{c\, r_1}\cos\varphi + \frac{2I'}{c\, r_2}\cos\varphi$$

$$H_{t2} = -\frac{2I''}{c\, r_1}\cos\varphi$$

$H_{t1} = H_{t2}$ $\qquad I'' = I - I'$

$$B_{n1} = \mu_1\frac{2I}{c\, r_1}\sin\varphi + \mu_1\frac{2I'}{c\, r_2}\sin\varphi$$

$$B_{n2} = \mu_2\frac{2I''}{c\, r_1}\sin\varphi$$

$B_{n1} = B_{n2}$ $\qquad \mu_2 I'' = \mu_1 I + \mu_1 I'$

Solution: $\qquad I' = I \left(\dfrac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \right)$

$$I'' = I \frac{2\mu_1}{\mu_2 + \mu_1}$$

Find all other fields from these values of I, I´, I´´

• Once any of **B**, **H**, or **M** are found in either region, the others can be found from the linear relations **B** = μ **H** and **B** = **H** + 4π **M**

Faraday's law of induction

So far only 'static' phenomena:

E(**r**) from static charge distribution ρ(**r**)

B(**r**) from time independent current **J**(**r**).

What happens if **J** = **J**(**r**,t) $\qquad$ **B** = **B**(**r**,t) ?

Faraday did these experiments.

• He was working as bookbinder's apprentice, educated himself by reading the books that were sent to the factory for binding. One of these was the Encyclopedia Brittanica.

• He wrote to Humphry Davy, Director of the Royal Institution, seeking scientific work. Davy saw promise in him, hired him as his assistant, and he became one of the pre-eminant scientists of the century, distinguishing himself in electrodynamics.

• He was a marvelous experimentalist and intuitive physicist, not educated in mathematics. His ideas on propagation of magnetic disturbances as waves were the inspiration for Maxwell to formulate his wave equations for electromagnetic fields.

• Faraday thought there should be an analogy between electricity and magnetism: an electric field induces a charge distribution on a conductor by its presence, and a magnet induces magnetism in an unmagnetised piece of iron. Shouldn't a current in one circuit induce a current in another circuit by its presence? He looked for such effects from the time of Oersted's discovery in 1819, recording his experiments in his notebook.

On Aug 29, 1831, by accident he noticed that a momentary current in one curcuit flowed when the switch completing a neighboring circuit was closed or opened. Induction was a *dynamic* effect.

Faraday found a current to be induced when

- current switched on or off in nearby circuit
- nearby circuit moved
- magnet moved nearby

All cause magnetic flux in circuit to change with time.

Experimental law of induction;

$$\int_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{1}{c}\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}\, da$$

$\mathbf{E}'$ is electric field in rest frame of circuit

$\frac{d}{dt}$ is total time derivative, due to motion of circuit or change in $\mathbf{B}$ in fixed circuit

$\int_S \mathbf{B}\cdot\mathbf{n}\, da$ = magnetic flux linking the circuit = Φ

- = Lenz's law, tells direction of current flow in circuit, always tends to oppose the change in flux

• Intimate connection between $\mathbf{E}$ and $\mathbf{B}$. Previously, each arose from different sources, with different vector character. Now, $\frac{d\mathbf{B}}{dt} \rightarrow \mathbf{E}$ without any stationary charge as source of **E**. Vector character of **E** from $\frac{d\mathbf{B}}{dt}$ is different from that produced by charge: $\frac{d\mathbf{B}}{dt}$ can be solenoidal. Connection leads to richly detailed new behavior, including relativity and the propagation of EM waves.

• No longer possible to speak separately of electricity and magnetism. They are one phenomena, electromagnetism.

• Name of **B** = magnetic induction field is not explained, since it is **B** and not **H** which causes induction

Examples of induction

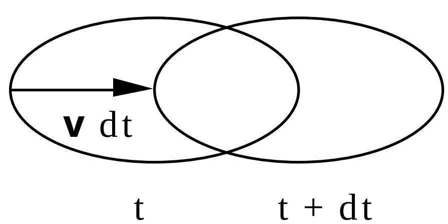
1) $\mathbf{B} = \mathbf{B}(t)$, spatially homogenous, circuit at rest.

$$\int_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{1}{c}\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da = -\frac{1}{c}\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, da = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

$$\mathbf{B}(t) = B_0 \cos \omega t \qquad \Phi = \int_S \mathbf{B} \cdot \mathbf{n}\, da = B_0 A \cos \omega t$$

$$V = -\frac{1}{c}\frac{\partial \Phi}{\partial t} = B_0 A\, \omega \sin \omega t$$

2) **B** is time independent, spatially inhomogenous



at time t, $\Phi(t) = \int_S \mathbf{B}(\mathbf{r}) \cdot \mathbf{n}\, da$

at time t + dt, circuit has moved **v** dt. Integrate flux over same area, bounded by same contour as at t, but use field values **v** dt displaced from circuit.

$$\Phi(t + dt) = \int_S \mathbf{B}(\mathbf{r} + \mathbf{v}\, dt) \cdot \mathbf{n}\, da$$

Taylor expansion of vector function: $\mathbf{B}(\mathbf{r} + \delta\mathbf{r}) = \mathbf{B}(\mathbf{r}) + \left(\delta\mathbf{r} \cdot \nabla\right) \mathbf{B}(\mathbf{r}) + \ldots$

proof: consider x component

$$B_x(\mathbf{r} + \delta\mathbf{r}) = B_x(\mathbf{r}) + \left.\frac{\partial B_x}{\partial x}\right|_{\mathbf{r}} \delta x + \left.\frac{\partial B_x}{\partial y}\right|_{\mathbf{r}} \delta y + \left.\frac{\partial B_x}{\partial z}\right|_{\mathbf{r}} \delta z + \ldots$$

$$= B_x(\mathbf{r}) + \delta\mathbf{r} \cdot \nabla B_x(\mathbf{r}) + \ldots$$

3 expansions, one for each component, can be written as a vector

$$\mathbf{B}(\mathbf{r} + \delta\mathbf{r}) = \mathbf{B}(\mathbf{r}) + \left(\delta\mathbf{r} \cdot \nabla\right) \mathbf{B}(\mathbf{r}) + \ldots$$

$$\Phi(t + dt) = \int_S \mathbf{B}(\mathbf{r} + \mathbf{v}\, dt) \cdot \mathbf{n}\, da = \int_S \mathbf{B}(\mathbf{r}) \cdot \mathbf{n}\, da + \int_S \left(\mathbf{v}\, dt \cdot \nabla\right) \mathbf{B}(\mathbf{r}) \cdot \mathbf{n}\, da$$

$$\frac{d\Phi}{dt} = \frac{\Phi(t + dt) - \Phi(t)}{dt} = \int_S \left(\mathbf{v} \cdot \nabla\right)\mathbf{B}(\mathbf{r}) \cdot \mathbf{n}\, da$$

Product rule: $\nabla \times \left(\mathbf{B} \times \mathbf{v}\right) = \mathbf{B} \underbrace{\left(\nabla \cdot \mathbf{v}\right)}_{0,\ \mathbf{v} = \text{const}} - \mathbf{v} \underbrace{\left(\nabla \cdot \mathbf{B}\right)}_{0,\ \text{no monopoles}} + \left(\mathbf{v} \cdot \nabla\right) \mathbf{B} - \underbrace{\left(\mathbf{B} \cdot \nabla\right) \mathbf{v}}_{0,\ \mathbf{v} = \text{const}}$

$$= \left(\mathbf{v} \cdot \nabla\right) \mathbf{B}$$

$$\frac{d\Phi}{dt} = \int_S \nabla \times \left(\mathbf{B} \times \mathbf{v}\right) \cdot \mathbf{n}\, da$$

$$= \int_C \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l}$$

Stokes theorem: $\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\mathbf{l} = \int_C \mathbf{F} \cdot d\mathbf{l}$

Faraday's law:

$$\int_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{1}{c} \int_C \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l}$$

or $\mathbf{E}' = -\frac{1}{c}\, \mathbf{B} \times \mathbf{v}$

↑ ↑

↑ field in lab

field in rest frame of circuit. This is field electrons in circuit feel.

- Circuit is gone. This is relation for point moving with velocity $\mathbf{v}$.

• Example of transformation of fields in moving frames, consistent with relativity.

Lab: $\mathbf{E} = 0$ $\mathbf{B} = \mathbf{B}$

Circuit: $\mathbf{E}' = \frac{1}{c}\,\mathbf{v} \times \mathbf{B}$ $\mathbf{B}' = \mathbf{B}$ (Compare Jackson 11.149)

These are correct for **B** **v**, in limit $\beta = \frac{\mathbf{v}}{c} << 1$. The reason for $\frac{\mathbf{v}}{c} << 1$ is that we assumed Galilean coordinate transformations in calculating $\Phi(t + dt)$; ie, we ignored the Lorentz contraction by the amount $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. Including this scaling factor, the fields would have been:

Lab: $\mathbf{E} = 0$ $\mathbf{B} = \mathbf{B}$

Circuit: $\mathbf{E}' = \frac{1}{c}\,\gamma\,\mathbf{v} \times \mathbf{B}$ $\mathbf{B}' = \gamma\,\mathbf{B}$

(Faraday's law does not tell what $\mathbf{B}'$ is; listed here for convenience)

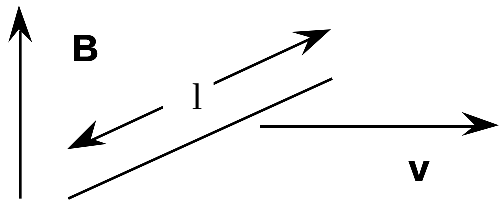
These are relativistically exact.

• Same result can be obtained from applying Lorentz force in moving wire. Each charge q feels a force $\mathbf{F} = q\,\frac{\mathbf{v}}{c} \times \mathbf{B} = q\,\frac{v\,B}{c}$ perpendicular to the wire and to **B**. This force is equivalent to and electric field $\mathbf{E}' = \frac{\mathbf{F}}{q} = \frac{\mathbf{v}}{c} \times \mathbf{B}$ on the charge in its rest frame, where $\mathbf{v} = 0$ and there is not magnetic force.

• This is the explanation of the Hall effect, where a transverse voltage appears in a current carrying wire **B** .

• Example of intimate connection between **E** and **B** : they tranform into each other simply by changing the reference frame. Pure **B** becomes **E** on frame of observation.

Can be used for situations where there is no complete circuit, e g wire moving through uniform field:



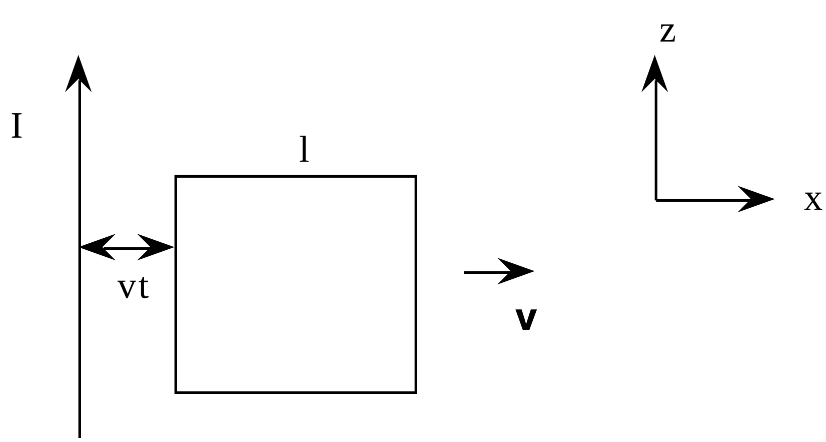
Lab: $\mathbf{E} = 0$ $\qquad$ $\mathbf{B} = \mathbf{B}$

Wire: $\mathbf{E}' = \frac{1}{c}\, \mathbf{v} \times \mathbf{B}$ $\qquad$ $\mathbf{B}' = \mathbf{B}$ $\qquad$ $\left(\text{for } \frac{\mathbf{v}}{c} \ll 1\right)$

Voltage across wire: $\Delta\varphi = \int_L \mathbf{E}' \cdot d\mathbf{l} = \frac{v\,B}{c}\, L$

Homework: consider cases where wire spins about its end.

Can also be used for complete circuits moving in non-uniform fields:



Homework: calculate by two methods

1) Find $\Phi(t)$ by integrating around the circuit at the instant t, differentiate to get $\frac{d\Phi}{dt}$

2) Use $\frac{d\Phi}{dt} = \int_C \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l}$

• There is a third way to consider this problem: the circuit is at rest, and the wire moves with velocity $-\mathbf{v}$. Same relative motion. Then $\mathbf{B}(\mathbf{r},t)$ is explicitly a function of time, and

$$\frac{d\Phi}{dt} = \int_C \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, da \qquad \text{integrated around a fixed contour}$$

This gives the same result as if the circuit is moved and the wire is at rest. The coincidence of these two apparently different physical situations stimulated Einstein to think about relative motion in Maxwell's equations. He uses it to open his paper published in 1905 "On the Electrodynamics of Moving Bodies" which introduced special relativity.

Maxwell's equations are invariant under Lorentz transformations, but not under Galilean transformations. Newton's laws are invariant under Galilean transformations, but not under Lorentz transformations. Although Newton's laws had been discovered much earlier and were more firmly established in the minds of physicists in 1905, Einstein saw that Lorentz transformations were the more acceptable, since they allowed light to have the same velocity in all inertial frames, and they allowed electrodynamics to be the same in all frames. He modified Newton's laws to be consistent with Lorentz transformations, a brilliant stroke of genius that has been verified many times in high energy accelerators, where $\frac{v}{c}$ can approach the speed of light.

Above gave differential form of Faraday's law for moving circuits. Now, differential form of Faraday's law for stationary circuits. This is the form normally quoted in Maxwell's equations.

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathbf{n}\, da \qquad \text{Stokes theorem: } \int_C \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot \mathbf{n}\, da$$

$$\int_S \left(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{n} \, da = 0$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \text{Faraday's law}$$

- $\nabla \times \mathbf{E} = 0$ is no longer correct, so $\mathbf{E} = -\nabla\varphi$ will no longer be correct. We will modify both φ and $\mathbf{A}$ to save the potentials later.

- Looks like **E** now has a new source. In analogy to the source equation for **B**,

$$\nabla \times \mathbf{E} = \frac{4\pi}{c}\left(-\frac{1}{4\pi}\frac{\partial \mathbf{B}}{\partial t} \right) \qquad \left(-\frac{1}{4\pi}\frac{\partial \mathbf{B}}{\partial t} \right) \text{plays the role of } \mathbf{J}$$

E arising from this source will be solenoidal not irrotational.

- No physical sources required for **E**: no charges or currents. A field $\frac{\partial \mathbf{B}}{\partial t}$ is now a source for another field **E**.

Energy in magnetic fields

In **E**, found energy by work done to assemble charge distribution from point charges at infinity. We found

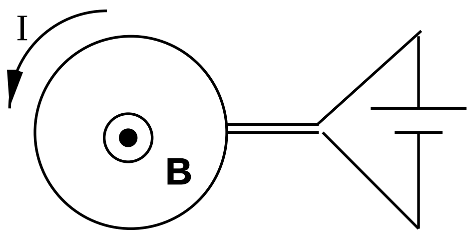
$$W = \frac{1}{2} \int_V \frac{1}{\kappa} \frac{\rho_{free}(\mathbf{r}')\, \rho_{free}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \, d^3r \, d^3r'$$

$$= \frac{1}{2} \int_V \varphi(\mathbf{r})\, \rho_{free}(\mathbf{r}) \, d^3r$$

$$= \frac{1}{8\pi} \int_V \mathbf{E} \cdot \mathbf{D} \, d^3r$$

Analagous method and expressions for magnetic energy. However, must turn on current in circuits to establish current distribution $\mathbf{J}(\mathbf{r})$, this involves changing fields and induction.

Circuit: work done/unit time is IV, where V is the inductive voltage from Faraday's law. Consider a single circular loop:



$$V = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{n}\,da$$

$$\mathbf{B}(\mathbf{r}) = \frac{I}{c}\oint_C \frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

Φ is linear in $\mathbf{B}$, $\mathbf{B}$ is linear in I

Define $\Phi = c\,L\,I$ L = self inductance, contains all geometry

$$\Phi = \int_S \mathbf{B}\cdot\mathbf{n}\,da = \frac{I}{c}\int_S\oint_C \frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\mathbf{n}\,da$$

$$L = \frac{1}{c^2}\int_S\oint_C \frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\mathbf{n}\,da \qquad \text{(usually not integrable)}$$

Then $V = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}\left(L\frac{dI}{dt}\right) = -L\frac{dI}{dt}$

As I rises from 0 to final value, inductive voltage $-L\frac{dI}{dt}$ appears around loop. By Lenz's law, it opposes the change in $\mathbf{B}$, thus it opposes I. Battery must do work $\frac{dW}{dt} = -IV = IL\frac{dI}{dt}$

In going from $I = 0$ to $I = I$, the work done is $\int dW = L\int I\frac{dI}{dt}\,dt = \frac{1}{2}LI^2$

This is the work stored in the field. It can be recovered by running the current down; then V is in the opposite direction, same direction as the current. The work done by the battery is negative, ie, it recovers the inductive work it did to create the field.

We do not include Joule heating, because this work is lost to heat and does not appear in the magnetic field.

3D distribution of current, keeping the geometry explicit:

$$\frac{dW}{dt} = -\int_V \mathbf{J}_{free}\cdot\mathbf{E}\,d^3r \qquad \mathbf{E}\text{ is the induced field in Faraday's law}$$

- If $\mathbf{J} \neq \mathbf{J}(t)$, then $\mathbf{B}$ is constant and there is no induced $\mathbf{E}$.
- Sign: increases in $\mathbf{J}$ always induce $\mathbf{E}$ in opposite direction (Lenz's law).

  Therefore $-\mathbf{J}\cdot\mathbf{E}$ is positive and is the work done by sources to create the increase in $\mathbf{J}$.

$$= -\int_V \frac{c}{4\pi}\nabla\times\mathbf{H}\cdot\mathbf{E}\,d^3r$$

$$= -\frac{c}{4\pi}\int_V \mathbf{H}\cdot\nabla\times\mathbf{E}\,d^3r \qquad \left(\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}\right)$$

$$= -\frac{c}{4\pi}\int_V \mathbf{H}\cdot\left(\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)d^3r \qquad \text{Faraday's law}$$

- For linear magnetic media, $\mathbf{B} = \mu\mathbf{H}$

$$\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{H}) = \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} + \mathbf{B}\cdot\frac{\partial\mathbf{H}}{\partial t} = \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} + \mu\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t} = \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H} + \mathbf{H}\cdot\frac{\partial\mu\mathbf{H}}{\partial t}$$

$$= 2\ \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{H}$$

$$\frac{dW}{dt} = \frac{1}{8\pi} \int_V \frac{\partial}{\partial t} \left(\mathbf{B} \cdot \mathbf{H} \right) d^3r = \frac{\partial}{\partial t} \left\{ \frac{1}{8\pi} \int_V \mathbf{B} \cdot \mathbf{H}\ d^3r \right\}$$

$$W = \frac{1}{8\pi} \int_V \mathbf{B} \cdot \mathbf{H}\ d^3r \qquad \text{integrate with respect to time}$$

- Constant of integration = 0 if no currents flow and no magnetic energy at t = 0.
- W in terms of fields, analgous to electrostatic expression $W = \frac{1}{8\pi} \int_V \mathbf{E} \cdot \mathbf{D}\ d^3r$

W in terms of potentials:

$$W = \frac{1}{8\pi} \int_V \nabla \times \mathbf{A} \cdot \mathbf{H}\ d^3r$$

$$= \frac{1}{8\pi} \int_V \mathbf{A} \cdot \nabla \times \mathbf{H}\ d^3r \qquad \left(\nabla \cdot \left(\mathbf{A} \times \mathbf{B} \right) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \right)$$

$$= \frac{1}{8\pi} \int_V \mathbf{A} \cdot \frac{4\pi}{c} \mathbf{J}_{\text{free}}\ d^3r = \frac{1}{2c} \int_V \mathbf{A} \cdot \mathbf{J}_{\text{free}}\ d^3r$$

- Analgous to electrostatic expression $W = \frac{1}{2} \int_V \varphi(\mathbf{r})\ \rho_{\text{free}}(\mathbf{r})\ d^3r$
- Need velocity prefactor to normalize out the implicity velocity in $\mathbf{J}$

In terms of sources:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int_V \frac{\mathbf{J}_{\text{free}}(\mathbf{r}') + \mathbf{J}_M(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\ d^3r'$$

- For linear media

$$\mathbf{J}_M = c\nabla \times \mathbf{M} = c\nabla \times \left(\chi \mathbf{H} \right) = c\ \chi\ \nabla \times \mathbf{H} = c\ \chi\ \frac{4\pi}{c} \mathbf{J}_{\text{free}} = 4\pi \chi\ \mathbf{J}_{\text{free}}$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int_V \frac{\mathbf{J}_{free}(\mathbf{r}') + 4\pi\chi\,\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\,d^3r' = \frac{1}{c}\int_V \frac{\mu\,\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\,d^3r'$$

$$W = \frac{1}{2c}\int_V \mathbf{A}\cdot\mathbf{J}_{free}\,d^3r = \frac{1}{2c}\int_V \frac{1}{c}\frac{\mu\,\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\cdot\mathbf{J}_{free}(\mathbf{r})\,d^3r'\,d^3r$$

$$= \frac{1}{2c^2}\int_V \frac{\mu\mathbf{J}_{free}(\mathbf{r}')\cdot\mathbf{J}_{free}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}\,d^3r'\,d^3r$$

- Analgous to electrostatic expression $W = \frac{1}{2}\int_V \frac{1}{\kappa}\frac{\rho_{free}(\mathbf{r}')\,\rho_{free}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}\,d^3r\,d^3r'$
- c^2 required in prefactor to normalize velocity units in $\mathbf{J}\cdot\mathbf{J}$
- Expressions contain effect of magnetic media through μ, dielectric media through κ. Only valid for linear polarizations in both cases.
- Permanent magnets:

$$\int_V \mathbf{B}\cdot\mathbf{H}\,d^3r = \int_V \nabla\times\mathbf{A}\cdot\mathbf{H}\,d^3r = \int_V \mathbf{A}\cdot\nabla\times\mathbf{H}\,d^3r = \int_V \mathbf{A}\cdot\frac{4\pi}{c}\mathbf{J}_{free}\,d^3r = 0$$

↑

integration by parts $\left(\nabla\cdot\left(\mathbf{A}\times\mathbf{B}\right) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}\right)$

Replace $\mathbf{M}$ by equivalent current: $\mathbf{J}_M = c\,\nabla\times\mathbf{M}$

W = \f(1,2c) \i(V , , $\mathbf{A}\cdot\mathbf{J}_M$ d^{3}r) = \f(1,2c) \i(V , , $\mathbf{A}\cdot c\nabla\times\mathbf{M}$ d^{3}r) =

$$\frac{1}{2}\int_V \nabla\times\mathbf{A}\cdot\mathbf{M}\,d^3r = \frac{1}{2}\int_V \mathbf{B}\cdot\mathbf{M}\,d^3r$$

Maxwell's equations

- Maxwell born in 1831, year Faraday discovered induction.
- From prominent Scottish family, father named Clerk, mother Maxwell.

• Excellent early education in letters, science, and mathematics at Edinburgh and later at Cambridge. First paper on EM in 1856 on translating Faraday's ideas on lines of force into mathematical language. 1856-60, Professor of Natural Philosophy at Marischal College, Aberdeen (he took this job because Scottish universities had 6 months off, which he could spend on his estate); 1860-65, at King's College, London, where he could communicate personally with Faraday, who had retired from the Royal Institution, London. Retired to his estate in Scotland in 1865 (age 34!) to write a complete account of the electromagnetic theory based on the four equations. 1871, returned to Cambridge as Professor of Experimental Physics (the greatest theorest of the time!). Became first director of Cavendish Laboratory, Cambridge. Died in 1879, at age 48. Worked also in kinetic theory of gases (Mawell-Boltzmann distribution), statistics, theory of rings of Saturn.

• Notable achievements in E&M:

corrected Ampere's law to include dynamic behavior
wrote the four laws (Maxwell's equations) in mathematical form
showed that EM disturbances could propagate as waves
suggested that light was an EM wave
replaced "action at a distance" with propagating disturbances in the fields

Displacement current

Equations based on experiment show certain asymmetries:

$\nabla \cdot \mathbf{D} = 4\pi\rho$	Coulomb's law
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}$	Ampere's law
$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$	Faraday's law
$\nabla \cdot \mathbf{B} = 0$	no magnetic monopoles

Faraday's law $\frac{\partial \mathbf{B}}{\partial t}$ creates $\mathbf{E}$. No such source for $\mathbf{B}$. No term describing $\frac{\partial \mathbf{E}}{\partial t}$.

Inconsistent with charge continuity:

$\nabla \cdot$ Ampere's law:

$$\frac{4\pi}{c} \nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{H}) = 0$$

But in general, $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$

Maxwell, in 1865, saw how to fix it.

$$\rho = \frac{1}{4\pi} \nabla \cdot \mathbf{D} \qquad\qquad \frac{\partial \rho}{\partial t} = \frac{1}{4\pi} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D})$$

Add $\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$ to Ampere's law

$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$ Then

$$0 = \nabla \cdot (\nabla \times \mathbf{H}) = \frac{4\pi}{c} \nabla \cdot \mathbf{J} + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) = \frac{4\pi}{c} \left\{ \nabla \cdot \mathbf{J} + \frac{1}{4\pi} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) \right\}$$

$$= \frac{4\pi}{c} \left\{ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right\}$$

$\frac{\partial \mathbf{D}}{\partial t}$ called "displacement current"

Maxwell's equations are consistent and complete:

$\nabla \cdot \mathbf{D} = 4\pi \rho$	Coulomb's law
$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}$	Ampere/Maxwell law (still called Ampere's law)
$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$	Faraday's law
$\nabla \cdot \mathbf{B} = 0$	no magnetic monopoles

• $\mathbf{H}$ has dynamic source $\frac{\partial \mathbf{D}}{\partial t}$ $\mathbf{E}$ has dynamic source $\frac{\partial \mathbf{B}}{\partial t}$

- $\mathbf{E}$ has both irrotational (ρ) and solenoidal $\left(\frac{\partial \mathbf{B}}{\partial t}\right)$ sources

  $\mathbf{B}$ has only solenoidal sources.
- Only asymmetries left are the absence of static magnetic monopoles as source for $\mathbf{B}$, and current of monopoles as source for $\mathbf{E}$. This is biggest unfinished question for electrodynamics: where are the monopoles? Principle of physics, everything which is not fobidden is mandatory, seems to be violated. Absence of monopoles is remarkable asymmetry of electrodynamics.

With Maxwell's addition, he showed that the fields satisfy a wave equation. In free space with no sources,

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

↑ ↑ ↑

∇ × Ampere's law ↑ Faraday's law

Maxwell's term

$$\nabla(\nabla \cdot \mathbf{B}) - {}^2\mathbf{B} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

↑ 0, no monopoles

$${}^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \qquad \text{wave equation for } \mathbf{B}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

↑ ↑ ↑

∇ × Faraday's law Ampere's law Maxwell's term

$$\nabla(\nabla \cdot \mathbf{E}) - {}^2\mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

↑ 0, no sources

$${}^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \text{wave equation for } \mathbf{E}$$

This equation was well known in 1865, as describing waves on water, in gases (sound) and in solids.

Fundamental property: solutions where two independent variables **r**, t are combined to a single variable x - ct or $\hat{\mathbf{k}} \cdot \mathbf{r}$ - ct, $\hat{\mathbf{k}}$ = propagation direction.

Example: $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \, f(\hat{\mathbf{k}} \cdot \mathbf{r} \pm ct)$

Suppose $\hat{\mathbf{k}}$ in z direction, $\hat{\mathbf{k}} \cdot \mathbf{r} = z$

$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \, f(z - ct)$ At t = 0, $\mathbf{E}(\mathbf{r}, 0) = \mathbf{E}_0 \, f(z)$

At t = t´, $\mathbf{E}(\mathbf{r}, t') = \mathbf{E}_0 \, f(z - ct')$

ie, contours of **E** have all moved by ct´ in time interval t´. Shape f(z) moves rigidly along z axis in time, with velocity c.

Solution to wave equation:

$$^2 \mathbf{E} = \ ^2 \mathbf{E}_0 \, f(z - ct) = \mathbf{E}_0 \frac{\partial^2}{\partial z^2} f(z - ct) = \mathbf{E}_0 \, f''$$

$$\frac{\partial^2}{\partial t^2} \mathbf{E} = \mathbf{E}_0 \frac{\partial^2}{\partial t^2} f(z - ct) = \mathbf{E}_0 \, f'' \, c^2$$

$$^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

• Normally f is a sin fcn, but it need not be

• Wave theories of light had been proposed many times before, as early as Huygens (1678) but it was never known what was oscillating to produce light. Upon finding the wave equation, Maxwell immediately saw that EM disturbances could propagate through space, and he suggested that light was such a propagating electromagnetic phenomenon, with **E** and **B** the quantities which were oscillating. This was a dramatically new idea: light was simply a manifestation of the ordinary electric and magnetic behavior which had been known for centuries. He was encouraged in this because the constant c was known from magnetism (3.1×10^8 m/sec), and the velocity of light was also approximately known from Fizeau's rotating toothed wheel experiment of 1849 (3.15×10^8 m/sec).

• The prediction that EM waves could be propagated through space was not confirmed until Hertz's experiments in 1887 in Karlsruhe. He was able to propagate waves across his laboratory, using a transmitter consisting of a single turn of wire with a gap across which he generated a spark, and a receiver an identical turn across which a spark jumped.

• This led to wireless telegraphy, consisting of a series of dots and dashes. The first commercial wireless telegraph message was sent in 1898 on the Isle of Wight, and Queen Victoria used wireless telegraphy regularly to communicate from the Isle of Wight with the Prince of Wales on the Royal Yacht. In 1899 a message was sent across the English Channel, 85 miles.

• Marconi was impressed with descriptions of Hertz's work in Italian journals. and he began to tinker in a laboratory he set up in his parents' villa. He was not trained in science, but his family was wealthy and sent him to England where he founded the British Marconi Company to communicate from ship to shore. In 1901 he sent the Morse code letter S (dot dot dot) across the Atlantic from Cornwall, England to St. John's, Newfoundland, 1700 miles.

• Vacuum tubes allowed amplification and the transmission of speech. An attempt to send wireless speech from Arlington, VA to the Eiffel Tower in 1915 was barely audible. It used more than 500 vacuum tubes, which kept burning out during the test.

• 1920: start of commercial radio broadcasts in the US.

• Waves in matter require an elastic medium to propagate: the elasticity provides a restoring force for the displaced particle to return to its equilibrium position. It was natural to assume a medium was needed for EM waves, to transmit the change in $\mathbf{E}$ or $\mathbf{B}$ to neighboring parts of space.

• The velocity of propagation would be c only with respect to this medium; if the medium moved with some velocity, the velocity of light would change by that amount. Analogy: sound waves in moving water or air. This gave rise to the concept of the ether drift, which was looked for in many careful experiments (Michelson and Morley, 1881-1897) but never found. The idea of the ether was not discarded until Einstein showed that assuming the velocity of light to be the same in all moving frames, and that the laws of physics (Maxwell's equations, Newton's laws) should be the same in all moving frames, removed inconsistencies in Mawell's equations for moving bodies, and made their predictions for the velocity of light agree with experiment (1905). This required only a re-interpretation of Maxwell's equations, which were correct as they stood, but a complete revision of Newtonian mechanics.

• With Maxwell, the concept of action at a distance was discarded.

Action at a distance: forces between objects which are not in contact, acting instantaneously. Gravity, Coulomb's law. If source of force moves, the force felt by other objects immediately adjusts to the new position of the source. Experiment will always confirm action at a distance if the measurement times are long compared to the propagation times.

Maxwell: changes in the source position or intensity are propagated with speed of light. If point charge moved, the change in the force on a distant object does not appear until $\Delta t = d/c$ later. Action is not instantaneous.

• Maxwell provided the primary theoretical driving force for EM. Previous progress was primarily driven by experiment, with theoretical description following, sometimes decades later. Maxwell put a theoretical insight into the equations, which had so many implications, that 25 years were required to work out the details and do experiments to confirm them. Enormous milestone for EM.

• Maxwell's equations were a crowning achievement of classical physics. They unified electricity, magnetism, and optics into one all inclusive mathematical form. Combined with Lorentz force law, $\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$, and Newton' law $\mathbf{F} = m\mathbf{a}$, they gave full quatitative description of dynamics of charged particles in EM fields. This was thought to describe all physical phenomena. In fact, it nearly does, provided only electronic effects are considered, distances are not too small, and velocities are not too fast. Electrons are governed by electrodynamics: Coulomb's law binds electrons to the nucleus, the "contact" between objects is electrostatic repulsion between their outer atomic electrons, chemistry is the exchange of atomic outer electrons, optics and light are propagation of EM waves, sound is mechanical vibration of air.

• Toward the end of the century, physics was considered to be a "closed science." Classical physics appeared capable of explaining everything, it was only a matter of working out the details. This left physicists with a satisfied feeling, to have achieved such knowledge, but also an empty feeling, because there were no new worlds to conquer.

• Myth of completeness was shattered in early 1900's, when Planck presented ideas on quanta which grew into revision of EM for small distances, and Einstein presented relativity, a complete revision of classical mechanics. Thomson discovered the electron in 1897, Rutherford discovered the nucleus in 1910, leading the way to new forces besides gravitation and EM, and ultimately to the discovery of new elementary particles which still goes on today.

• For EM, a major challenge was to resolve the idea of light quanta, introduced by Einstein in 1905 to explain the photoelectric effect, with Maxwell's wave theory of light. Waves can be made as small as possible in amplitude and energy, but for a given frequency, the smallest quanta of light has energy $h\nu$. Ultimately, the quantum picture of light is correct and the wave theory is incorrect, but if many quanta are involved, they mimic the wave properties perfectly. Thus, the great triumph of Maxwell, that light could be quantitatively described as a wave propagating in an ether, had to yield to relativity and quantum mechanics, basic revisions of physical understanding inspired by new experimental discoveries.