PHY 2061 Exam 2 Solutions

Problem 1 (relativity, 10 points):

Protons leave a particle accelerator with a speed of **v = 0.9c** to enter and travel through an evacuated tube **1.0 m** long, as measured by an observer in the laboratory, and then finally reach a detector. (Note: $c = 3.0\text{x}10^8$m/s)

Part A (2 points):

How long does it take for a proton to travel from one end of the tube to the other (**in nanoseconds**) according to an observer in the laboratory frame?

Answer: 3.7

Solution: In any frame distance equals speed times time and hence

$$t = \frac{d}{v} = \frac{1m}{0.9(3\times 10^8 m/s)} = 3.7\times 10^{-9}s = 3.7ns\,.$$

Part B (3 points):

How long does it take for a proton to travel from one end of the tube to the other (**in nanoseconds**) according to an observer traveling with a proton?

Answer: 1.6

Solution: The proton frame is the proper time and hence

$$t' = \tau = \frac{t}{\gamma} = \frac{3.7ns}{2.294} = 1.6ns\,,$$

where I used $\gamma = 1/\sqrt{1-\beta^2} = 2.294$.

Part C (3 points):

What is the tube's length (**in m**) as measured by an observer traveling with a proton?

Answer: 0.436

Solution: In any frame distance equals speed times time and hence

$$d' = v\tau = \frac{vt}{\gamma} = \frac{d}{\gamma} = \frac{1m}{2.294} = 0.436m\,.$$

Part D (2 points):

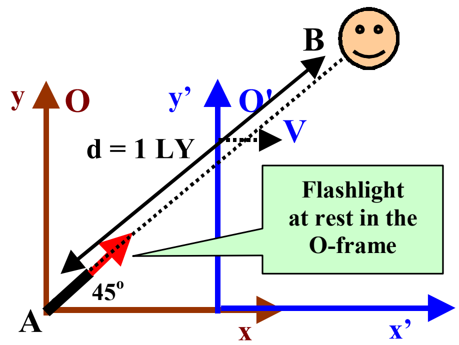
What is the kinetic energy of the protons (**in MeV**) in the laboratory frame? (Note: The rest mass of a proton is M_pc^2 = 940 MeV)?

Answer: 1,216

Solution: The kinetic energy is given by

$$KE = (\gamma - 1)M_pc^2 = 1.294(940MeV) = 1{,}216MeV\,.$$

Problem 2 (relativity, 12 points):



A flashlight at rest at the origin (**A**) of the **O-frame** is orientated at a **45°** angle with the **x-axis** as shown in the **Figure**. At **t = 0** the flashlight emits a light signal toward an observer (**B**) located a distance **d = 1 light-year** away (as measured in the **O-frame**). The **O'-frame** which is moving along the x-axis of the **O-frame** with speed $V = c/\sqrt{2}$ and at **t = t' = 0** the origins of the two frames coincide. (Note: A light-year is a measure of distance equal to 9.46×10^{15} m, which is the distance light travels in one year.)

Part A (3 points):

What are the x' and y' components of the velocity of the light signal in the **O'-frame**? (*i.e.* what are $\mathbf{v_x'}$ and $\mathbf{v_y'}$? Express your answer in terms of c.)

Answer: $v'_x = 0$ $v'_y = c$

Solution: The components of the velocity of the light signal in the O-frame are given by

$$v_x = \cos 45° c = c/\sqrt{2}$$
$$v_y = \sin 45° c = c/\sqrt{2}$$

The components of the velocity of the light signal in the O'-frame are given by

$$v'_x = \frac{v_x - V}{1 - V v_x / c^2} = 0$$

$$v'_y = \frac{v_y}{\gamma(1 - V v_x / c^2)} = \frac{c/\sqrt{2}}{\gamma(1-\beta^2)} = \frac{\gamma c}{\sqrt{2}} = c$$

where I have used $\beta = c/\sqrt{2} = V/c = v_x/c$ and $\gamma = 1/\sqrt{1-\beta^2} = \sqrt{2}$.

Part B (2 points):

How long (**in years**) does it take the light signal to get from **A** to **B** according to an observer in the **O-frame**?

Answer: 1

Solution: In any frame distance equals speed times time and hence

$$t = \frac{d}{v} = \frac{(1y)c}{c} = 1y.$$

Part C (3 points):

How long (**in years**) does it take the light signal to get from **A** to **B** according to an observer in the **O'-frame**?

Answer: 0.707

Solution: The time in the O'-frame is given by

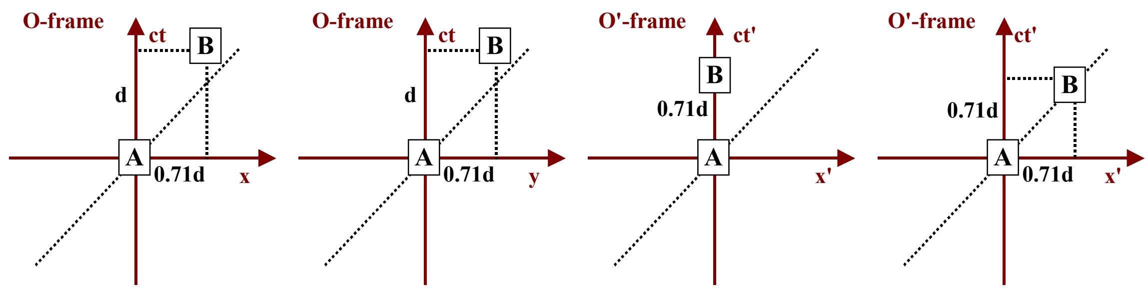
$$c\Delta t' = \gamma(c\Delta t - \beta \Delta x) = \gamma(d - \beta d/\sqrt{2}) = \tfrac{1}{2}\gamma d = d/\sqrt{2},$$

where I used $\Delta x = d/\sqrt{2}$ and $\gamma = 1/\sqrt{1-\beta^2} = \sqrt{2}$. Hence,

$$\Delta t' = \frac{d}{\sqrt{2}c} = \frac{(1y)c}{\sqrt{2}c} = 0.707y$$

Part D (4 points):

Let **event A** be the sending of the light signal at **A** and **event B** be the receiving of the light signal at **B**. Plot and label **event A** and **event B** on the following four space-time plots:



O-frame

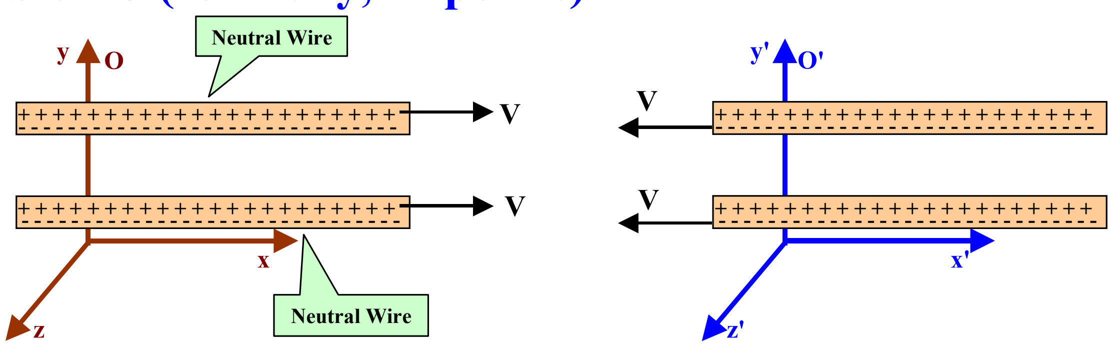
$$c\Delta t = d$$
$$\Delta x = d/\sqrt{2}$$
$$\Delta y = d/\sqrt{2}$$

with d = 1 L-Y.

O'-frame

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x) = d/\sqrt{2}$$
$$\Delta x' = \gamma(\Delta x - \beta c\Delta t) = 0$$
$$\Delta y' = \Delta y = d/\sqrt{2}$$

Problem 3 (relativity, 12 points):



In the lab frame (**O-frame**) there are two identical parallel **neutral** wires consisting of a line of positive charges moving to the right with speed V = 0.75c and a line of negative charges at rest as shown in the figure. The **O'-frame** is moving the right at speed V = 0.75c relative to the O-frame. In the O'-frame the positive charges are at rest and the negative charges are moving to the left at speed V = 0.75c. The rest density of the positive charges is **1 μC/m**.

Part A (3 points):

What is the current carried by one of the wires in the **lab frame** (**in A**)?

Answer: 340

Solution: The current in the **lab frame** is given by

$$I = \lambda^+ V = \gamma\lambda_0^+ V = 1.512(1\times10^{-6}\,C/m)0.75(3\times10^8\,m/s) = 340A,$$

where I used $\gamma = 1/\sqrt{1-\beta^2} = 1.512$. Note that the wires are neutral in the lab frame which implies that

$$\lambda = \lambda^+ - \lambda^- = \gamma\lambda_0^+ - \lambda_0^- = 0,$$

and hence

$$\lambda_0^- = \gamma\lambda_0^+,$$

where λ_0^+ and λ_0^- are the rest densities of positives and negatives, respectively.

Part B (2 points):

Which of the following statements is true in the **lab frame**? (*circle all correct statements*)

(a) There is an electric force attracting the two wires.
(b) There is an electric force repelling the two wires.
(c) There is a magnetic force attracting the two wires.
(d) There is a magnetic force repelling the two wires.
(e) There are no forces on the wires.

Part C (3 points):

What is the *net* charge density of one of the wires in the **O'-frame** (**in μC/m**)?

Answer: -1.286

Solution: The charge density of the wires in the **O'-frame** is given by

$$\lambda' = \lambda'^+ - \lambda'^- = \lambda_0^+ - \gamma\lambda_0^- = (1-\gamma^2)\lambda_0^+ = -\beta^2\gamma^2\lambda_0^+$$
$$= -(0.75)^2(1.512)^2(1\mu C/m) = -1.286\mu C/m$$

Note that the wires are negatively charged.

Part D (2 points):

What is the current carried by one of the wires in the **O'-frame** (**in A**)?

Answer: 514

Solution: The current in the **O'-frame** is given by

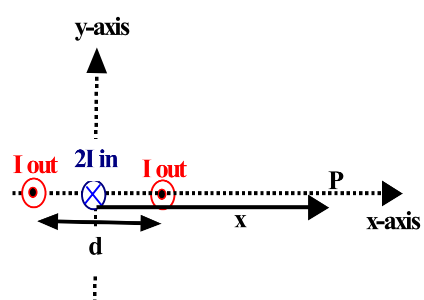
$$I' = \lambda'^- V = \gamma\lambda_0^- V = \gamma^2\lambda_0^+ V = \gamma I = 1.512(340A) = 514A$$

Part E (2 points):

Which of the following statements is true in the **O'-frame**? (*circle all correct statements*)

(a) There is an electric force attracting the two wires.
(b) There is an electric force repelling the two wires.
(c) There is a magnetic force attracting the two wires.
(d) There is a magnetic force repelling the two wires.
(e) There are no forces on the wires.

Problem 4 (10 points):



Three infinitely long parallel wires lie in the xz-plane and are located at **x = -d/2**, **0**, and **d/2** as shown in the **Figure**. The two outside wires (x=-d/2, d/2) carry equal currents **I** (*out of page*), while the center wire (x = 0) caries a current of **2I** (*into the page*).

Part A (4 points):

What is the magnetic **field vector,** $\vec{B}(x)$, on the x-axis a distance **x > d/2** from the origin? (Express your answer in terms of k = $\mu_0/4\pi$, x, I, and d.)

Answer: $\vec{B}(x) = \dfrac{kId^2}{x(x^2-(d/2)^2)}\hat{y}$

Solution: The magnetic field at **P** is the superposition of the fields from each of the three wires as follows:

$$\vec{B}(x) = \left(\frac{2kI}{x+d/2} + \frac{2kI}{x-d/2} - \frac{2k(2I)}{x} \right) \hat{y} = \frac{kId^2}{x(x^2-(d/2)^2)} \hat{y}.$$

Part B (3 points):

Which of the following gives the magnitude of the magnetic field on the x-axis at **x = d**? (*circle one answer and show your work*)

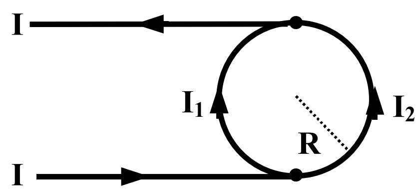
(a) 4kI/d	(b) 8kI/d	$\boxed{\text{(c) 4kI/(3d)}}$	(d) 2kI/d
(e) kI/d	(f) 3kI/(4d)	(g) zero	(h) none

Part C (3 points):

Which of the following gives the magnitude of the magnetic field on the x-axis in the limit **x >> d**? (*circle one answer and show your work*)

(a) $4kId/x^2$	$\boxed{\text{(b) } kId^2/x^3}$	(c) $2kId^3/x^4$	(d) $2kId/x^2$
(e) $2kI/x$	(f) $4kI/x$	(g) $2kId^2/x^3$	(h) none

Problem 5 (12 points):



Two parallel semi-infinite thin wires each carry a current **I**, which is split into two semi-circles with radius **R**, as shown in the **Figure**. The current $\mathbf{I_1}$ goes through the left semi-circle and current $\mathbf{I_2}$ through the right semi-circle ($\mathbf{I = I_1 + I_2}$), where the fraction of the total current **I** going through the left semi-circle is $\mathbf{f = I_1/I}$ ($0 < f < 1$). (Note: $k = \mu_0/4\pi = 10^{-7}$ Tm/A)

Part A (3 points):

If the z-axis is out of the paper, which of the following gives the z-component of the magnetic field at the center of the circle? (*circle one answer and show your work*)

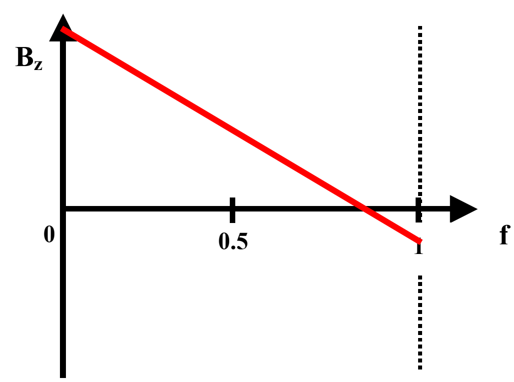
(a) $\pi k(1-2f)I/R$	$\boxed{\text{(b) } kI(2+\pi(1-2f))/R}$	(c) $\pi k(2f-1)I/R$	(d) $kI(\pi(1-2f)-2)/R$
(e) $2\pi kI/R$	(f) $2\pi k(1-2f)I/R$	(g) zero	(h) none

Solution: The magnetic field at center of the circle is the superposition of the fields from each of the two semi-circles and each of the two semi-infinite wires as follows:

$$B_z = \frac{kI}{R} + \frac{kI}{R} + \frac{\pi k I_2}{R} - \frac{\pi k I_1}{R} = \frac{2kI}{R} + \frac{\pi k(I_2 - I_1)}{R} = \frac{k(2+\pi(1-2f))I}{R}.$$

Part B (3 points):

Sketch the z-component of the magnetic field at the center of the circle as a function of the fraction $\mathbf{f = I_1/I}$ from **f = 0** to **f = 1**.



Part C (3 points):
For what value of the fraction **f** is the net magnetic field zero at the center of the circle?

Answer: 0.818

Solution: From **Part A** we see that the magnetic field is zero when

$$2+\pi(1-2f)=0 \quad \text{or} \quad f=\frac{1}{2}+\frac{1}{\pi}=0.818.$$

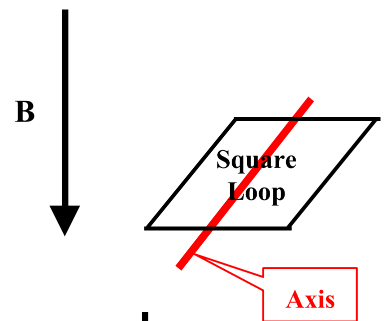
Part D (3 points):
If **R = 3.14 m**, **I = 2 A**, and **f = ¼**, what is the magnitude of the magnetic field at the center of the circle (**in microT**)?

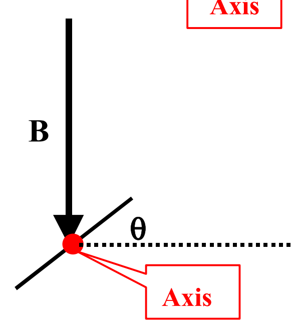
Answer: 0.227

Solution: From Part A we have

$$B_z=\frac{k(2+\pi(1-2f))I}{R}=\frac{(10^{-7}Tm/A)(2+0.5\pi)(2A)}{3.14m}=0.227\mu T.$$

Problem 6 (10 points):
A single wire square loop with resistance **R = 0.3 Ω** and area **A = 100 cm^2** rotates with a constant angular velocity in a uniform **2 T** magnetic field. The axis of rotation goes through the center of the loop (in the plane of the loop) and is perpendicular to the magnetic field as shown in the **Figure**. The loop makes **10 revolutions per second**.





Part A (4 points):
What is the maximum induced current generated in the loop (**in Amps**)?

Answer: 4.19

Solution: The magnetic flux through the loop is given by **Φ_B = BAcosθ** and the induced **EMF** is,

$$\varepsilon=-\frac{d\Phi_B}{dt}=-BA\frac{d\cos\theta}{dt}=BA\,\omega\sin\omega t\,,$$

where **θ = ωt** and **ω = dθ/dt** is the angular velocity. The current is given by

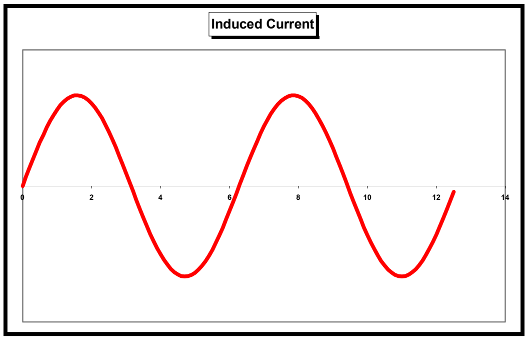
$$I=\frac{\varepsilon}{R}=\frac{BA\,\omega}{R}\sin\omega t=I_{\max}\sin\omega t\,.$$

The maximum current occurs when **sinθ = 1** and thus

$$I_{\max}=\frac{BA\,\omega}{R}=\frac{2\pi BA}{RT}=\frac{2\pi(2T)(100\times10^{-4}m^2)}{(0.3\Omega)(0.1s)}=4.19\,A\,,$$

where I used **T = 2π/ω =1/f = 0.1 s**.

Part B (2 points):
Sketch the induced current **I(t)** in the loop as a function of time.



Part C (4 points):

How much energy (**in Joules**) is dissipated in the wire due to heating in one complete revolution? (Identities: $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$)

Answer: 0.26

Solution: The power dissipated by the resistance in the wire loop is

$$P = \frac{dU}{dt} = I^2 R = I_{\max}^2 R \sin^2 \omega t \text{ ,}$$

and hence

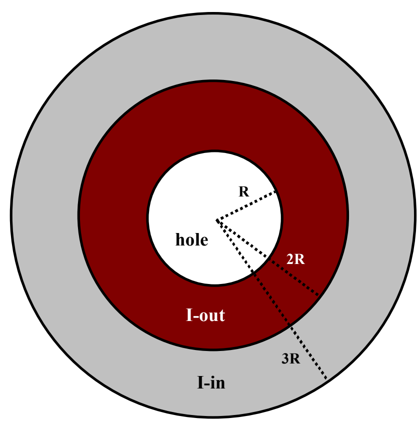
$$U = \int P dt = \int_0^T I_{\max}^2 R \sin^2 \omega t dt = \frac{I_{\max}^2 R}{\omega} \int_0^{2\pi} \sin^2 \theta d\theta$$

$$= \frac{I_{\max}^2 RT}{2} = \frac{(4.19A)^2 (0.3\Omega)(0.1s)}{2} = 0.26J$$

where I used

$$\int_0^{2\pi} \sin^2 \theta d\theta = \int_0^{2\pi} \tfrac{1}{2}(1-\cos 2\theta) d\theta = \pi \text{ .}$$

Problem 7 (12 points):



An infinitely long coaxial cable consists of an inner wire of radius **2R** carrying a uniformly distributed current **I** (*out of the page*) surrounded by an outer wire with inner radius **2R** and outer radius **3R** carrying a uniformly distributed current **I** (*in to the page*). There is a cyclinderical hole with radius R in the inner wire as shown in the **Figure**. For both wires the current density is constant. (Express your answers in terms of k, I, and R.)

Part A (2 points):

What is the magnitude of the magnetic field outside of the coaxial cable a distance **r** from the center (**r > 3R**)?

Answer: zero

Solution: Ampere's Law tell us that

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

and if we take a circular loop with radius **r > 3R** then $\mathbf{I_{enclosed}} = \mathbf{0}$ and hence **B = 0**.

Part B (3 points):

What is the magnitude of the magnetic field inside the outer wire a distance **r** from the center (**2R < r < 3R**)?

Answer: $B(r) = \dfrac{2kI}{r}\left(\dfrac{9R^2 - r^2}{5R^2}\right)$

Solution: If we take a counter-clockwise circular loop with radius **r** such that **2R < r < 3R** then

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B(r) = \mu_0 I_{enclosed} = \mu_0 I - \mu_0 J_1(\pi r^2 - \pi(2R)^2) = \mu_0 \frac{9R^2 - r^2}{5R^2} I ,$$

where I used

$$I = J_1(\pi(3R)^2 - \pi(2R)^2) = J_1 \pi 5R^2 ,$$

and where $\mathbf{J_1}$ is the current density (constant) of the outer wire. Thus,

$$B(r) = \frac{2kI}{r}\left(\frac{9R^2 - r^2}{5R^2}\right).$$

Part C (3 points):

What is the magnitude of the magnetic field inside the inner wire a distance **r** from the center (**R < r < 2R**)?

Answer: $B(r) = \dfrac{2kI}{r}\left(\dfrac{r^2 - R^2}{3R^2}\right)$

Solution: If we take a counter-clockwise circular loop with radius **r** such that **R < r < 2R** then

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B(r) = \mu_0 I_{enclosed} = \mu_0 J_1(\pi r^2 - \pi R^2) = \mu_0 \frac{r^2 - R^2}{3R^2} I ,$$

where I used

$$I = J_2(\pi(2R)^2 - \pi R^2) = J_2 \pi 3R^2 ,$$

and where $\mathbf{J_1}$ is the current density (constant) of the outer wire. Thus,

$$B(r) = \frac{2kI}{r}\left(\frac{r^2 - R^2}{3R^2}\right).$$

Part D (2 points):

What is the magnitude of the magnetic field inside the hole a distance **r** from the center (**r < R**)?

Answer: zero

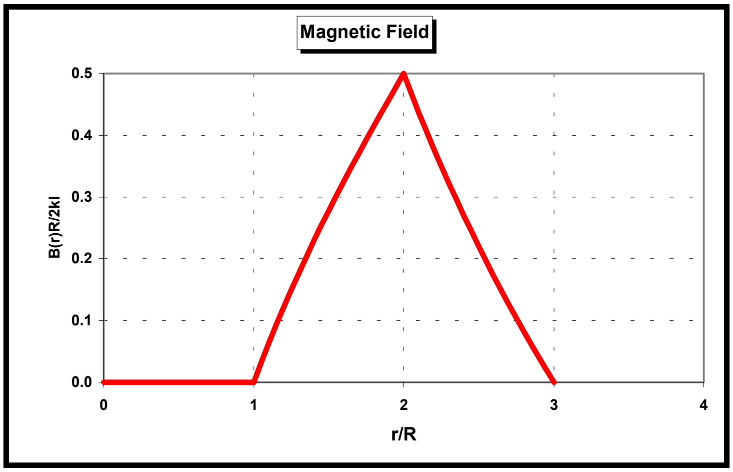
Solution: Ampere's Law tell us that

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

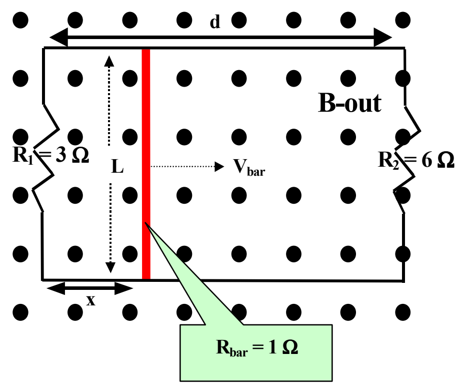
and if we take a circular loop with radius **r < R** then $\mathbf{I_{enclosed}} = \mathbf{0}$ and hence **B = 0**.

Part E (2 points):

Plot the magnitude of the magnetic field as a function of **r** from **r = 0** to **r = 3R**.



Problem 8 (10 points):



A moveable (massless and frictionless) bar with a length of **L = 3 meters** and resistance $\mathbf{R_{bar} = 1\ \Omega}$ is being moved to the right along two conducting rails at a constant speed $\mathbf{V_{bar} = 4\ m/s}$ as shown in the **Figure**. The rails and the two resistors, $\mathbf{R_1 = 3\ \Omega}$ on the left and $\mathbf{R_2 = 6\ \Omega}$ on the right form a rectangle with a width **d = 12 meters**. The entire system is immersed in a uniform **3T** magnetic field (*out of the paper*). The bar is at x = 0 at t = 0 (*i.e.* at resistor R_1).

Part A (2 points):

What is the magnitude of the induced current (**in Amps**) in the **3 Ohm** resistor at **t = 1 second**?

Answer: 8

Solution: There are two loops which have a changing magnetic flux. If I choose my orientation counter-clockwise then the magnetic flux through the loop consisting of R_1 and the bar is given by

$$\Phi_B = B_z xL = B_z VLt \ ,$$

and from Faraday's Law of induction the induced EMF is

$$\varepsilon_1 = -\frac{d\Phi_B}{dt} = -B_z VL \ ,$$

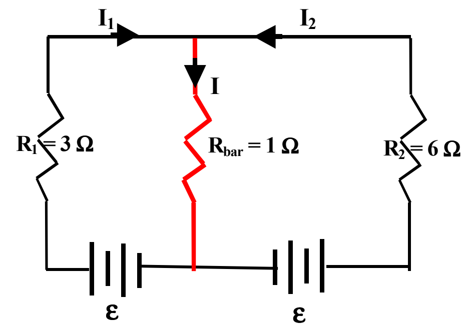
If I choose my orientation counter-clockwise then the magnetic flux through the loop consisting of R_2 and the bar is given by

$$\Phi_B = B_z(d - x)L = B_z(d - Vt)L \ ,$$

and from Faraday's Law of induction the induced EMF is

$$\varepsilon_2 = -\frac{d\Phi_B}{dt} = B_z VL \ ,$$

Thus, we have a circuit that is equivalent to the following:



where $\varepsilon = B_z V L = (3T)(4m/s)(3m) = 36$ **V**. Using the usual circuit rules we see that

$$\varepsilon - I_1 R_1 - I R_{bar} = 0$$
$$\varepsilon - I_2 R_2 - I R_{bar} = 0$$
$$I_1 R_1 = I_2 R_2$$

Solving for $\mathbf{I = I_1 + I_2}$ yields

$$I = \frac{\varepsilon}{(R_{eff} + R_{bar})} = \frac{60V}{(2\Omega + 1\Omega)} = 12A$$

where $R_{eff} = \frac{R_1 R_2}{R_1 + R_2} = \frac{(3\Omega)(6\Omega)}{(3\Omega + 6\Omega)} = 2\Omega$ and $I_1 = \frac{R_2}{(R_1 + R_2)} = \frac{6}{9}(12A) = 8A$.

Part B (2 points):

What is the magnitude of the induced current (**in Amps**) in the **6 Ohm** resistor at **t = 1 second**?

Answer: 4

Solution: From **Part A** we see that

$$I_2 = \frac{R_1}{(R_1 + R_2)} I = \frac{3}{9}(12A) = 4A.$$

Part C (2 points):

What is the magnitude of the induced current (**in Amps**) in the **1 Ohm** bar at **t = 1 second**?

Answer: 12

Solution: From **Part A** we see that

$$I = \frac{\varepsilon}{(R_{eff} + R_{bar})} = \frac{60V}{(2\Omega + 1\Omega)} = 12A.$$

Part C (4 points):

How much work must be done (**in Joules**) to move the bar at the constant speed of **4 m/s** from **x = 0** to **x = 12 m**?

Answer: 1,296

Solution: The work is given by

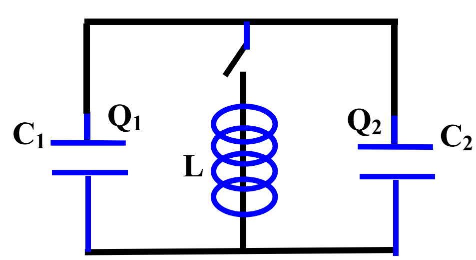
$$W = F_{ext} d = ILB_z d = (12A)(3m)(3T)(12m) = 1{,}296J,$$

where I used $\vec{F}_{ext} = -\vec{F}_B = -I\vec{L} \times \vec{B}$. Note that this is the same as

$$U = Pt = (I_1^2 R_1 + I_2^2 R_2 + I^2 R_{bar})t$$
$$= ((8A)^2(3\Omega) + (4A)^2(6\Omega) + (12A)^2(1\Omega))(3s) = 1{,}296J$$

where **t = d/V = (12m)/(4m/s) = 3 s.**

Problem 9 (12 points):



An inductor **L** and two capacitors (in parallel) $\mathbf{C_1}$ and $\mathbf{C_2}$ form an **LC circuit** as shown in the **Figure**. The capacitors have a total initial charge of $\mathbf{Q_0 = Q_1 + Q_2 = 5}$ **microC** and $\mathbf{C_1 = 2}$ **microF**, $\mathbf{C_2 = 3}$ **microF**, and **L = 0.5 milliH**. The switch is closed at **t = 0**.

Part A (2 points):

If the system in in equilibrium, how much charge (**in μC**) is on each capacitor before the switch is closed?

Answer: Q_1(before) = 2 μC Q_2(before) = 3 μC

Solution: The potential drop across each resistor is equal so before the switch is closed we have

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \text{ and } Q_1 + Q_2 = Q_0 .$$

Solving for $\mathbf{Q_1}$ and $\mathbf{Q_2}$ yields

$$Q_1 = \frac{C_1}{C_1 + C_2} Q_0 = \frac{2}{5}(5\mu C) = 2\mu C$$
$$Q_2 = \frac{C_2}{C_1 + C_2} Q_0 = \frac{3}{5}(5\mu C) = 3\mu C$$

Part B (3 points):

What is the initial stored energy in the system (**in microJ**)?

Answer: 2.5

Solution: The initial stored energy is given by

$$U_i = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} = \frac{Q_0^2}{2C_{eff}} = \frac{Q_0^2}{2(C_1 + C_2)} = \frac{(5\mu C)^2}{2(2\mu F + 3\mu F)} = 2.5\mu J$$

Part C (3 points):

After the switch is closed, what is the frequency of the subsequent oscillations (**in kHz**)?

Answer: 3.18

Solution: The angular frequency is given by

$$\omega = \frac{1}{\sqrt{LC_{eff}}} = \frac{1}{\sqrt{L(C_1 + C_2)}} = \frac{1}{\sqrt{(0.5mH)(5\mu F)}} = 2\times 10^4 / s ,$$

and the linear frequency is

$$f = \frac{\omega}{2\pi} = \frac{2\times 10^4 / s}{2\pi} = 3.18\,kHz .$$

Part D (4 points):

After the switch is closed, what is the maximum current through the inductor (**in Amps**)?

Answer: 0.1

Solution: Energy conservation tell us that

$$\frac{1}{2}LI^2 + \frac{Q^2}{2C_{eff}} = U_i = \frac{Q_0^2}{2C_{eff}} \text{ and hence } \frac{1}{2}LI_{\max}^2 = U_i = \frac{Q_0^2}{2C_{eff}} .$$

Thus,

$$I_{\max} = \frac{Q_0}{\sqrt{LC_{eff}}} = \omega Q_0 = (2\times 10^4 / s)(5\mu C) = 0.1A .$$