# Electromagnetism
# &
# Electromagnetic Waves

## (Lecture notes)

**by**

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**The information contained in this booklet,**

- **Is meant to supplement material covered in lectures,**
- **And is not intended to replace the recommended textbook for the course on electromagnetism.**
- **<u>Lectures may also contain additional material and problem solving that will be directly related to the examinations.</u>**

## 1.1 Introduction and Vector field theory

In order to study the subject of electromagnetism and to appreciate the beauty of its mathematical description it is necessary to be familiar with the mathematical tools that must be used to express rigorously its various basic laws. Such laws are simple in form and, in the main, will already be familiar to most level 2 students. Indeed, many students are already familiar with most of the basic laws before they leave school. Here we will express such laws in an elegant mathematical form and investigate the consequences and implications of these formulations, particularly with regard to electromagnetic radiation and its propagation through space.

Before beginning our studies of electromagnetism it is useful to remind ourselves of the mathematical theories associated with vector fields. Without these one cannot really appreciate the subject since a qualitative visualisation of the physics of three-dimensional time varying fields can only be taken so far, even by the most imaginative minds. We therefore begin with a recap of vector field theory. In the space and time available this can only be an outline. Those students to whom this subject is completely new are recommended to read an appropriate book on the subject such as Feynmann's lectures on physics volume II.

## 1.2 Back to Basics

The electromagnetic force is the "glue" holding us all together and is responsible for the way atoms and molecules are constructed. It is interesting to note that if two people had 10% of their electrons removed from them and were standing at arms length the force between them would be sufficient to lift the weight of the planet earth in its own gravitational field!

The force between two static charges $Q_1$ and $Q_2$ is an electrostatic force and is given by Coulombs law

$$\underline{F} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}\hat{\underline{r}} \tag{1.1}$$

where r is the distance between the charges and $\varepsilon_0$ ( = 8.85x10$^{-12}$ F/m) is the permittivity of free space. $\hat{\underline{r}}$ is a unit vector in the direction of r. [What are the S.I. units of $Q_1$, r and $\underline{F}$?] If the two charges are of the same sign, e.g both electrons or both protons, the force is one of repulsion. If they are of opposite sign then the force is attractive.

By supposing the existence of a quantity known as the electric field $\underline{E}$, and defining it as the force per unit charge on a very small positive test charge placed at the point of interest, we may write the force on a charge Q in an electric field as

$$\underline{F} = Q\underline{E} \tag{1.2}$$

It also follows that the electric field around a point charge Q is given by

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2}\hat{\underline{r}} \tag{1.3}$$

If such a charge Q experiences an additional force when it is travelling through space with a velocity $\underline{v}$ then we assert that there exists an additional field $\underline{B}$ called the magnetic field. Consequently, if there are both types of field at some point in space the net force on the moving charge will be

$$\underline{F} = Q\left(\underline{E} + \underline{v} \wedge \underline{B}\right) \tag{1.4}$$

In such a situation the force F is a vector sum of the electrostatic force, given by equation (2) and a magnetically induced force called the Lorentz force, equal to $Q\underline{v} \wedge \underline{B}$.

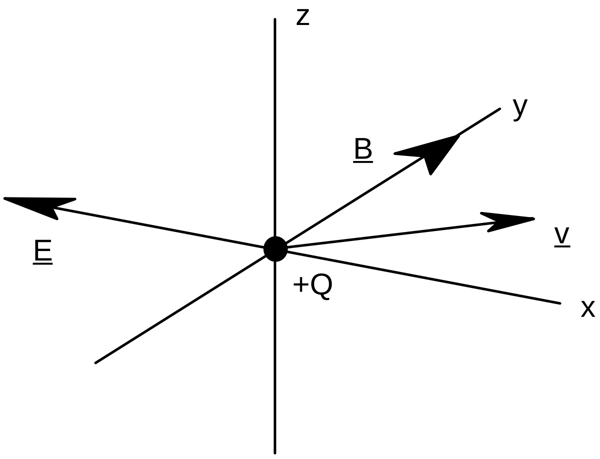



Fig 1

On the diagram across, showing a charge Q moving with velocity v in fields E and B, draw in the electrostatic force, the Lorentz force and the resultant force. (What would happen if the charge were negative?)

The electric and magnetic fields E and B may, of course, vary throughout space and may also vary with time. Providing we know how E and B vary with the co-ordinates x, y, z and t we can, in principle, determine the ultimate motion of the charged particle. However, such a computation may not necessarily be simple!

It is these fields and others that are the subject of our studies. As you will know E and B can be produced by electrostatic charges and circulating currents respectively. Can you think of any other sources of electric and magnetic fields?

## Principle of Superposition

Suppose a set of charges (or a single charge) produces a field $\underline{E}_1$ in space. And now suppose another set of charges produces a field $\underline{E}_2$ at the same point in space. If both sets of charges are in place at the same time the resultant field $\underline{E}$ is the linear superposition of the two fields $\underline{E}_1$and $\underline{E}_2$. Thus

$$\underline{E} = \underline{E}_1 + \underline{E}_2 \tag{1.5}$$

Likewise, in the case of the magnetic field we might have

$$\underline{B} = \underline{B}_1 + \underline{B}_2 \tag{1.6}$$

Although the presence of these fields in space may be detected by the forces experienced by a static and/or moving test charge, their presence is assumed even in the absence of this charge. The magnitude and direction of E and B can be measured in terms of the force on such a test charge. However, the test charge must be very small so that we may assume that it does not distort those fields that we are trying to measure.

We now associate every point (x, y, z) in space as possibly having an electric or magnetic field that may vary with time t. We could write these fields as

$$\underline{E}(x,y,z,t) \quad \text{and} \quad \underline{B}(x,y,z,t) \tag{1.7}$$

## 1.3 Visualisation Problem

One of the major problems in electromagnetism is that the quantities that we are working with, namely E and B, cannot be directly seen or felt. This often makes it difficult to imagine what is going on in the space around charges or magnets or current carrying conductors. However, Michael Faraday got a long way in this subject with his concept of lines of force. We too can take advantage of this useful way of picturing vector fields.

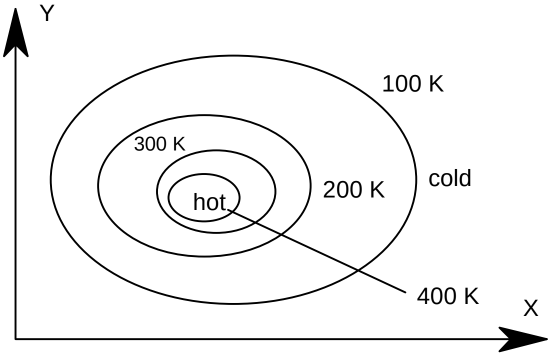



Fig 2

Of course, fields may be of scalar quantities {e.g Temperature. Fig 2 shows a set of isothermals (lines of constant temperature)

In 3D these contours would be surfaces over which the temperature is constant.

## 1.4 Vector Fields

If the quantity has magnitude and direction the field is a vector field. (e.g "Velocity Field" where the velocity is that of a flowing liquid or gas)

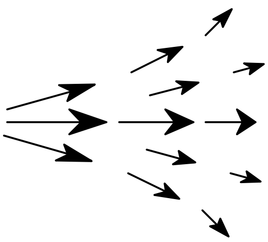


Fig 3

Arrows points in the direction of "flow: The size of the arrow indicates how fast the liquid is moving at a position in space

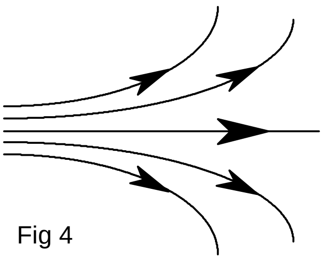


Fig 4

Here the tangent to the line gives the direction of "flow." and the number of lines/unit area gives the magnitude of the velocity.

The above ideas also apply to the vector fields $\underline{E}$ and $\underline{B}$ even though the "flow" doesn't have the same physical interpretation as that of moving liquid.

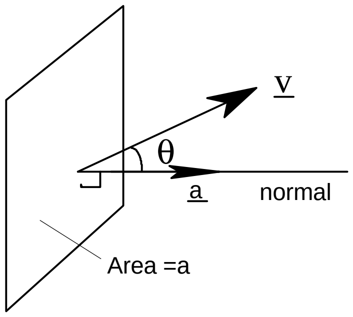
## 1.5 Characteristics of Vector Fields

There are two mathematically important properties of a vector field that are necessary in our description of the laws of electromagnetism.

## FLUX

Imagine a closed surface in a vector field and ask; are we losing something from inside the surface" or more specifically "does more fluid flow out than flows in through the surface per unit time? We call the net outflow of fluid through the whole surface per second the **FLUX** or flux of velocity.

The flow through an element of area of the surface is the normal component of velocity $\underline{v}$ times the area, assuming that v is constant over the surface area a. This is shown in figure 5.




Flux = $\underline{v} \cdot \underline{a}$

$= v\, a \cos\theta$

Fig 5

For a closed surface we might need to do an integration of this flux over the entire surface or perhaps write

FLUX = (Av normal component of $\underline{v}$) x (total surface area)

These arguments about flux also apply to other vectors, including $\underline{E}$ and $\underline{B}$ and heat flow $\underline{h}$. Heat flow is a good one to visualise since one can imagine a generator of heat at the centre of a block with energy "heat flux" flowing out of the surface of the block as shown in figure 6.

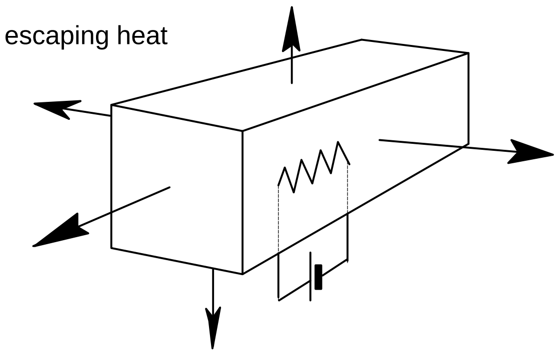



Fig 6

In the case of $\underline{E}$ and $\underline{B}$ it is not quite so easy to visualise the flow of anything. Nevertheless we still speak of the flux of $\underline{E}$ or $\underline{B}$ and not only through a **closed** surface but any bounded surface.

In figure 7 we see electric field lines passing out of a closed spherical surface. If the electric field is constant over the surface and given by

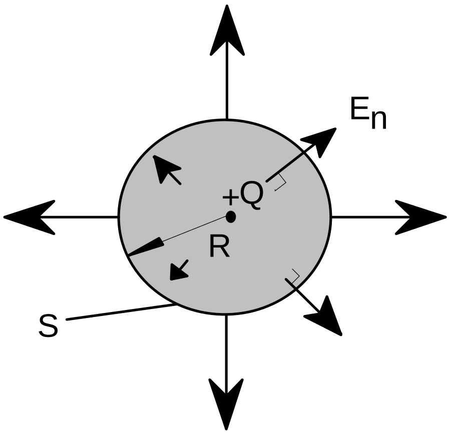



Fig 7

$$E = E_n$$

then the flux through the surface S is

$$\text{Flux of E} = E_n 4\pi R^2$$

## CIRCULATION

There is a second property of a vector field that has to do with a line rather than a surface.

Imagine a velocity field describing the flow of a liquid. We might ask the question: "Is the liquid circulating"? By this we mean, is there a net rotational motion around some closed loop? See figure 8 and consider the "eddy".

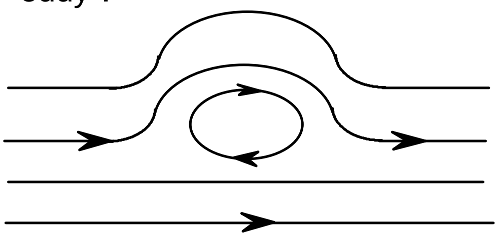


Fig 8

If we sailed around the eddy we would experience that the velocity vector is always pointing in the same sense ("clockwise").

The **circulation** around any imagined closed curve is defined as the average tangential component of the vector (in a consistent sense) multiplied (or integrated around) by the circumference of the loop.

Circulation = (av Tangential component) x (distance around)

Circulation also applies to the vectors $\underline{E}$ and $\underline{B}$ although, AGAIN, it is difficult to visualise what it is that is "flowing around the loop".

With these two ideas - **FLUX** and **CIRCULATION** we can describe all the laws of electricity and magnetism.

## 1.6 The Laws of Electromagnetism (Free Space)

You may not understand them immediately but here are the basic laws of electromagnetism in words.

1. The net flux of E though any closed surface = $\dfrac{\text{Total charge inside}}{\varepsilon_0}$

where $\varepsilon_0 = 8.85 \times 10^{12}$ F / m permittivity of free space.

**This is Gauss's law**

2. The circulation of E around a loop $= -\dfrac{d}{dt}(\text{Flux of B through S})$

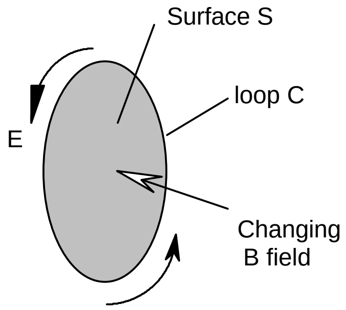



Fig 9

**This is Faraday's law**

3. The Flux of B through any closed surface = ZERO

**(No isolated magnetic poles).**

4. Circulation of B around a loop = flux of current density through S x $\mu_0$

$$+\varepsilon_0\mu_0 \frac{d}{dt}(\text{flux of E through } S)$$

**(Amperes law + new term.)**

NOTE:

$$\left.\begin{array}{l} dB/dt \rightarrow E \\ dE/dt \rightarrow B \end{array}\right) \rightarrow \text{EM radiation}$$

and $c = 1/\sqrt{\varepsilon_0\mu_0}$ this is the velocity of light in vacuum.

**J C Maxwell's equations were the most significant scientific event of the 19th century.**

## 1.7 Differential Calculus of Vector Fields

We must now begin an abstract, mathematical view of electromagnetism in order to give substance to the laws just given. But to do this we must first explain a new and peculiar notation we will want to use - the mathematics of vector fields. (Not just important for $\underline{E}$ and $\underline{B}$ but applies to all vector fields and therefore many kinds of physical situations).

First it will be assumed that you know what a vector is and that you are aware of the following identities:

$$\underline{A} \,.\, \underline{B} = \text{scalar} = A_xB_x + A_yB_y + A_zB_z == |A||B|\text{Cos}\theta \tag{1.8}$$

$$\underline{A} \wedge \underline{B} = \underline{C} \tag{1.9}$$

where

$$C_x = A_yB_z - A_zB_y$$

$$C_y = A_z B_x - A_x B_z$$
$$C_z = A_x B_y - A_y B_x$$

$$\underline{A} \wedge \underline{A} = 0 \qquad (1.10)$$

$$\underline{A} \cdot (\underline{A} \wedge \underline{B}) = 0 \qquad (1.11)$$

$$\underline{A} \cdot (\underline{B} \wedge \underline{C}) = (\underline{A} \wedge \underline{B}) \cdot \underline{C} \qquad (1.12)$$

$$\underline{A} \wedge \underline{B} \wedge \underline{C} = \underline{B}(\underline{A} \cdot \underline{C}) - \underline{C}(\underline{A} \cdot \underline{B}) \qquad (1.13)$$

Also

$$df(x,y,z) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z \qquad (1.14)$$

as $\delta x, \delta y, \delta z \rightarrow 0$

and
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad (1.15)$$

## Scalar and Vector Fields T, h

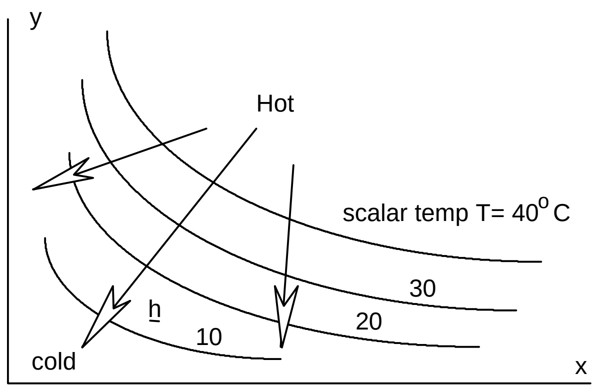



Fig 10

In figure 10 we see a set of isothermals representing a scalar field. Superimposed on this we see a vector field i.e. that of heat flow. You will see that heat flows from the hot zone to the cold zone.

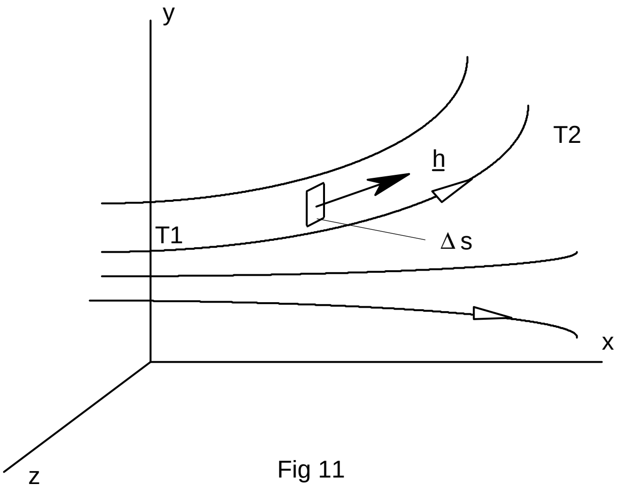



Fig 11

In figure 11 we see only heat flow lines. The vector $\underline{h}$ represents the magnitude and direction of the flow of heat energy per unit area per sec at various points in space. The heat flows in the direction of the h vector.

In figure 12 the thermal energy flowing, in the direction of h, through the elemental areas $\Delta S_2$ and $\Delta S_1$ is the same and is equal to $\Delta Q$

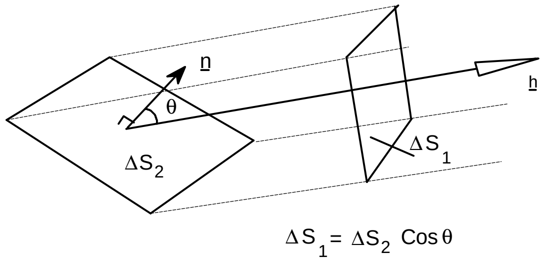


Fig 12

The heat flow/unit area through $\Delta S_1$

$$= \ h \ = \ \frac{\Delta Q}{\Delta S_1}$$

The heat flow/unit area through $\Delta S_2$

$$= \ \underline{h} \,.\, \underline{\hat{n}} \ = \ \frac{\Delta Q}{\Delta S_2} \ = \ \frac{\Delta Q}{\Delta S_1} \text{Cos}\,\theta$$

Note in above $\Delta Q$ is the flux of $\underline{h}$ through both areas.

i.e. Flux of h is $\Delta Q \ = \ \underline{h} \,.\, \underline{\Delta S_2}$.
(The vector $\underline{\Delta S_2}$ points in the direction of the normal $\underline{n}$ to the plane containing $\Delta S_2$)

## 1.8 Derivatives of Fields - The Gradient

For a scalar field say of temp T we may consider a derivative with respect to position

e.g $\quad \left(\frac{\partial T}{\delta x}, \ \frac{\partial T}{\partial y}, \ \frac{\partial T}{\partial z}\right)$

Do these three differentials form a vector?

We know

$$\underline{A} \,.\, \underline{B} \ = \ \text{scalar}$$

so that if $\underline{A}$ is a vector then so is $\underline{B}$.

Consider,

$$T \ = \ f(x,y,z)$$

The temperature difference between two nearby points is:

$$\delta T \ = \ \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z \qquad (1.16)$$

i.e. a scalar

Now, since the position vector is

$$\underline{\delta \ell} \ = \ \underset{\sim}{i}\,\delta x + \underset{\sim}{j}\,\delta y + \underset{\sim}{k}\,\delta z$$

Then

$$\underset{\sim}{i}\frac{\partial T}{\partial x} + \underset{\sim}{j}\frac{\partial T}{\partial y} + \underset{\sim}{k}\frac{\partial T}{\partial z} \qquad \textbf{must be a vector}$$

This special vector is called grad T or gradient of T and is usually written

$$\text{grad}T \ = \ \nabla T \ = \ \underset{\sim}{i}\frac{\partial T}{\partial x} + \underset{\sim}{j}\frac{\partial T}{\partial y} + \underset{\sim}{k}\frac{\partial T}{\partial z} \qquad (1.17)$$

$$\therefore \qquad \delta T = \nabla T . \underline{\delta \ell} \qquad (1.18)$$

($\nabla T$ is actually the temp gradient and it depends on direction.)

$\nabla$ **is called the "DEL" operator.**

$$\nabla \equiv \left(\nabla_x, \nabla_y, \nabla_z\right) \equiv \underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z} \qquad (1.19)$$

and is an operator hungry for something to "operate" on. On its own it has no meaning.

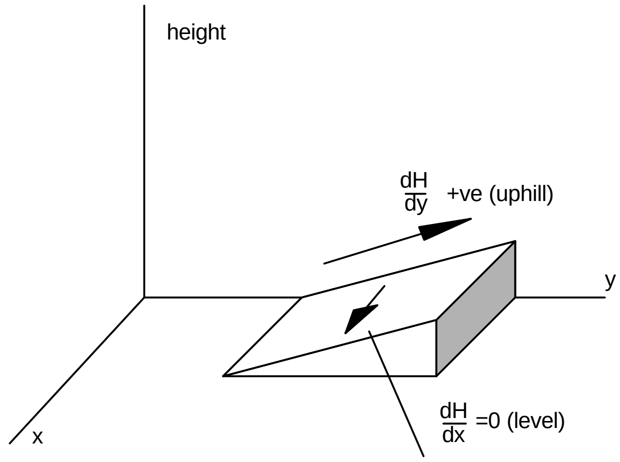



Fig 13

In figure 13 we see a gradient of a familiar scalar quantity, height (H). We are familiar with gradients: up-hill and down-hill. In the figure we see that in one direction Grad(height) is positive (up-hill). At right angles to this the gradient is zero.

## Operations with $\underline{\nabla}$ -The Divergence

What else can we do with the operator $\underline{\nabla}$?

For example what does $\nabla . \underline{h} =$ scalar mean?

It is actually given by
$$\nabla . \underline{h} = \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} \qquad (1.20)$$

And we give it the name **"Divergence of h"**

Sometimes written **Div $\underline{\mathbf{h}}$ = $\nabla$ . $\underline{\mathbf{h}}$**

We will look at its physical meaning later.

## The Curl of a Vector

Now what about

$\nabla \wedge \underline{h}$ what does this mean?

The x-component of it is
$$\left(\nabla \wedge \underline{h}\right)_x = \left(\frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z}\right) \qquad (1.21)$$

Fill in the rest

$$\left(\nabla \wedge \underline{h}\right)_x = \qquad\qquad \left(\nabla \wedge \underline{h}\right)_z =$$

This operation leads to a vector result and has the special name:

$$\text{Curl } \underline{h} = \nabla \wedge \underline{h} = \text{vector} \tag{1.22}$$

The basic laws of electromagnetism i.e. - Maxwell's Equations are now within our grasp!

## Maxwell's Equations (our pretty equations!)

They are written as follows for **FREE SPACE:**

| | | |
|---|---|---|
| 1. $\nabla \cdot \underline{E} = \rho / \varepsilon_0$ | Gauss's law (differential form) | (1.23) |
| 2. $\nabla \wedge \underline{E} = \dfrac{-\partial \underline{B}}{\partial t}$ | Faraday's law (differential form) | (1.24) |
| 3. $\nabla \cdot \underline{B} = 0$ | No magnetic monopoles | (1.25) |
| 4. $\nabla \wedge \underline{B} = \mu_0 \underline{J} + \varepsilon_0 \mu_0 \dfrac{\partial \underline{E}}{\partial t}$ | Ampere's law (differential form) | (1.26) |

where $\underline{J}$ = current density (A/m$^2$) and $\rho$ = charge density (C/m$^3$)

## Second Derivatives of Fields

Before continuing, we look at some other operations with "DEL"

i) $\nabla \cdot \nabla T = \nabla^2 T$ = a scalar field (1.27)

ii) $\nabla \wedge \nabla T = 0$ (1.28)

iii) $\nabla \cdot \left(\nabla \cdot \underline{h}\right)$ = vector field (1.29)

iv) $\nabla \cdot \left(\nabla \wedge \underline{h}\right) = 0$ (1.30)

v) $\nabla \wedge \left(\nabla \wedge \underline{h}\right) = \nabla \cdot \left(\nabla \cdot \underline{h}\right) - \nabla^2 \underline{h}$ (1.31)

vi) $\left(\nabla \cdot \underline{h}\right)\underline{h} = \nabla^2 \underline{h}$ = vector field (1.32)

## 1.8 Vector Integral Calculus

We have already discussed the meaning of the gradient of a scalar $\nabla T$ and this is relatively easily understood. We now turn our attention to the other derivative operations, the Divergence and Curl of a vector. An appreciation of the meaning of these quantities may be obtained through integral expressions.

## Line Integral of $\nabla$T (grad T)

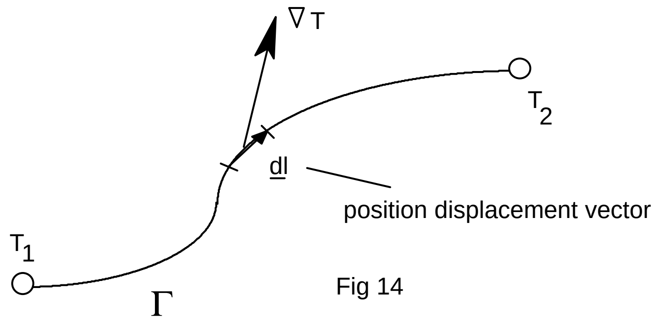



Fig 14

If we carryout a line integral along the curve $\Gamma$ from (1) to (2) we obtain

$$\int_{T_1}^{T_2} dT \;=\; T_2 - T_1 \;=\; \int_1^2 \nabla T.\underline{d\ell}$$

along $\Gamma$

Note the difference in temp $T_2 - T_1$ is a scalar quantity, hence the value of the integral on the RHS above, is independent of the curve $\Gamma$ (any path may be taken).

## The Flux of a Vector Field *"An understanding of Divergence"*

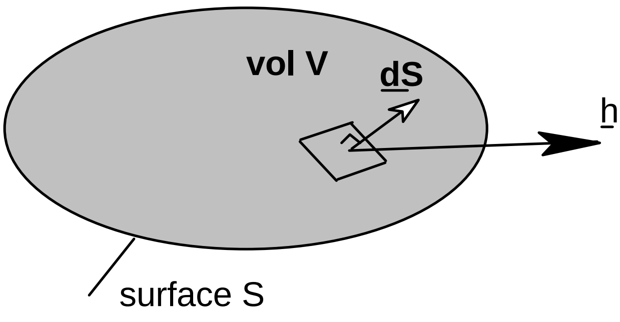



Fig 15

Suppose we have heat flow out of a volume V through the enclosing surface S. The flux of $\underline{h}$ through S is written

$$\int_S \underline{h}.\underline{dS} \;=\; \text{total heat flow}$$

where $\underline{dS}$ is an elemental vector whose magnitude equals the area and whose direction is normal to the surface.

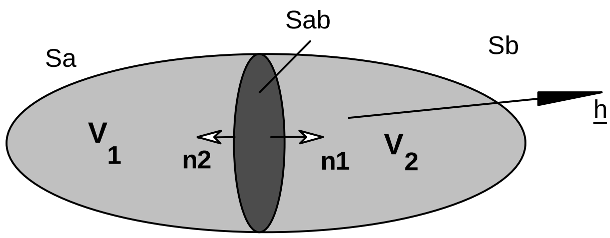



Fig 16

Now we split this volume into two and consider each of the two sections separately.
These are shown in figure 16 as volume $V_1$ and $V_2$ with corresponding surfaces $S_1$ and $S_2$. These two surface areas are made up as follows

$$S_1 \;=\; S_a + S_{ab} \qquad \text{and} \qquad S_2 \;=\; S_b + S_{ab}$$

Consequently ,the Flux through $\quad S_1 \;=\; \int_{S_a} \underline{h}\,.\,\underline{dS} + \int_{S_{ab}\to n_1} \underline{h}\,.\,\underline{dS}$

and the Flux through $\quad S_2 \;=\; \int_{S_b} \underline{h}\,.\,\underline{dS} + \int_{S_{ab}\to n_2} \underline{h}\,.\,\underline{dS}$

However, we have $\quad \int_{S_{ab}\to n_1} \underline{h}\,.\,\underline{dS} \;=\; -\int_{S_{ab}\to n_2} \underline{h}\,.\,\underline{dS}$

Hence, the Total flux through $\quad S_1 + S_2 \;=\; \int_{S_a} \underline{h}\,.\,\underline{ds} + \int_{S_b} \underline{h}\,.\,\underline{ds} \;=$ flux through S

Therefore we see that the flux out of the surface of the whole volume is equal to the sum of fluxes through the surfaces of smaller volumes obtained by dividing the original into two or more sub-volumes.

**Flux from a Cube**

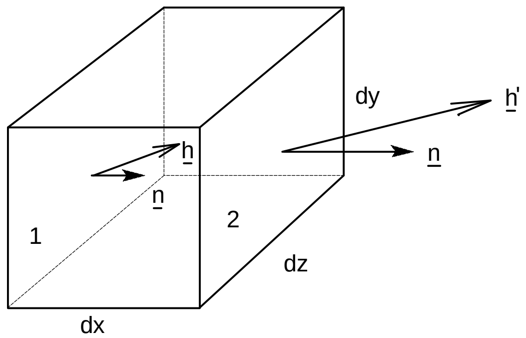



Fig 17

Now let us look at the flux of a vector quantity $\underline{h}$ which leaves the surface of an elemental cube as shown in figure 17.

Flux of h OUT of face 1

$$F_{1x} \;=\; -h_x \delta y \delta z \quad \text{elemental cube}$$

Flux out of 2

$$F_{2x} \;=\; h_x' \delta y \delta z$$

However, $\quad h_x' \;=\; h_x + \dfrac{\delta h_x}{\delta x}\delta x$

Hence the flux out of 2 is

$$\left(h_x + \frac{\delta h_x}{\delta x}\delta x\right)\delta y \delta z$$

And the Net Flux out of faces 1 and 2

$$F_x \;=\; F_{2x} + F_{1x} \;=\; \frac{\delta h_x}{\delta x}\delta x \delta y \delta z$$

Similarly $\quad F_y \;=\; \dfrac{\delta h_y}{\delta y}\delta x \delta y \delta z$

and $\quad F_z \;=\; \dfrac{\delta h_z}{\delta z}\delta x \delta y \delta z$

$\therefore$ The Total Flux out of the cube is

$$\int_{\text{Cube}} \underline{h} \,.\, \underline{dS} \;=\; \left(\frac{dh_x}{dx} + \frac{dh_y}{dy} + \frac{dh_z}{dz}\right) dV \qquad (1.33)$$

where $dV \;=\; dxdydz$ is the elemental volume of the cube in the limit as $\delta V \to dV$
Equation (1.33) may now be written

$$\int_{\text{Cube}} \underline{h} \,.\, \underline{dS} \;=\; \nabla \,.\, \underline{h}\, dV \qquad (1.34)$$

We now begin to see the meaning of $\nabla$ **.** $\underline{\mathbf{h}}$ at some point in space, **it is the outgoing flow or flux of $\underline{\mathbf{h}}$ per unit volume.**

(c.f. density at a point that is defined as mass/unit volume).

Since any volume V can be made up from an infinitely large number of very small elemental volumes we may write

$$\int_S \underline{h} \cdot \underline{dS} = \int_V \nabla \cdot \underline{h} \; dV \tag{1.35}$$

This is **Gauss's Theorem** and Applies to all vectors

for example $$\int_V \nabla \cdot \underline{E} \, dV = \int_S \underline{E} \cdot \underline{dS}$$

V and S are macroscopic volumes and corresponding surfaces.

## The Circulation of a Vector Field

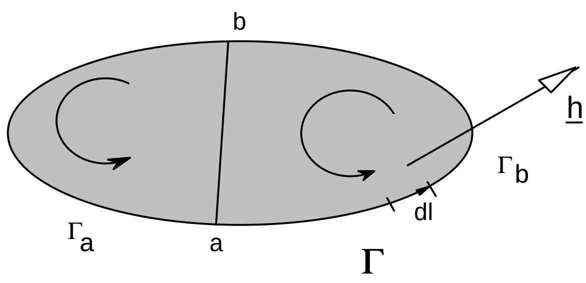



Fig 18

We now look at the circulation (or swirling) of vector fields.

Consider the line integral of $\underline{h}$ around the closed loop $\Gamma$ in figure 18.

$\oint_\Gamma \underline{h} \cdot \underline{d\ell} \Rightarrow$ integral of Tangential component of $\underline{h} \cdot \underline{d\ell}$ around loop $\Gamma$

Now we split the loop into two sub-loops as shown, where,

LH Loop $\quad \Gamma_1 = \Gamma_a + \Gamma_{ab}$

RH Loop $\quad \Gamma_2 = \Gamma_b + \Gamma_{ba}$

hence, we may write,

$$\oint_{\Gamma_1} \underline{h} \cdot \underline{d\ell} = \int_{\Gamma_a} \underline{h} \cdot \underline{d\ell} + \int_{\Gamma_{ab}} \underline{h} \cdot \underline{d\ell}$$

and

$$\oint_{\Gamma_2} \underline{h} \cdot \underline{d\ell} = \int_{\Gamma_b} \underline{h} \cdot \underline{d\ell} + \int_{\Gamma_{ba}} \underline{h} \cdot \underline{d\ell}$$

Now since $$\int_{\Gamma_{ab}} \underline{h} \cdot \underline{d\ell} = -\int_{\Gamma_{ba}} \underline{h} \cdot \underline{d\ell}$$

we have $$\oint_{\Gamma_1} \underline{h} \cdot \underline{d\ell} + \oint_{\Gamma_2} \underline{h} \cdot \underline{d\ell} = \int_{\Gamma a} \underline{h} \cdot \underline{d\ell} + \int_{\Gamma_b} \underline{h} \cdot \underline{d\ell} = \oint_{\Gamma} \underline{h} \cdot \underline{d\ell}$$

*See if you can put this equation into words. It's about circulations around minor loops being equal to circulation around a major loop. (refer to figure 19)*

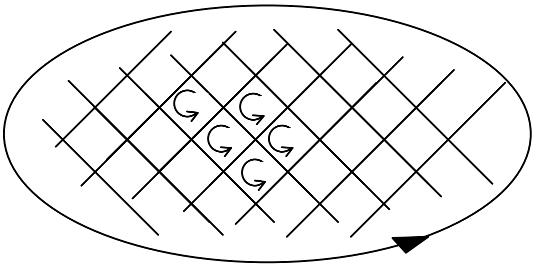



Fig 19

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## **Circulation around a Square**

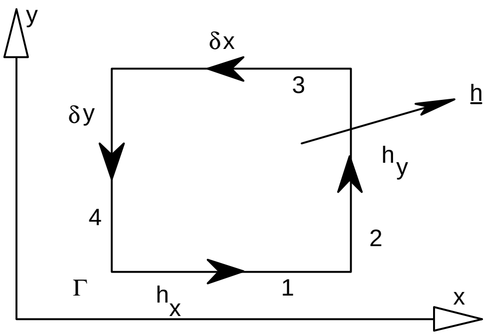


Fig 20

Now lets have a look at the circulation of a vector around a square loop as shown in figure 20.

We see that,

$$\oint_\Gamma \underline{h} \cdot \underline{d\ell} = h_{x1}\delta x + h_{y2}\delta y - h_{x3}\delta x - h_{y4}\delta y$$

$$= (h_{x1} - h_{x3})\delta x + (h_{y2} - h_{y4})\delta y$$

But $\quad h_{x3} = h_{x1} + \frac{dh_x}{dy}\delta y \quad$ and

$$h_{y2} = h_{y4} + \frac{dh_y}{dx}\delta x$$

$$\therefore \qquad \oint_\Gamma \underline{h} \cdot \underline{d\ell} = \left(\frac{dh_y}{dx} - \frac{dh_x}{dy}\right)dxdy$$

The term in brackets is the z-component of Curl $\underline{h}$ so we may write the above equation

$$\oint_\Gamma \underline{h} \cdot \underline{d\ell} = \left(\nabla \wedge \underline{h}\right)_z dS$$

$\left(\nabla \wedge \underline{h}\right)_z$ is the component of $\nabla \wedge \underline{h}$ which is normal to the elemental surface dS,

so we can write this

$$\oint_\Gamma \underline{h} \cdot \underline{d\ell} = \left(\nabla \wedge \underline{h}\right)_n dS$$

or

$$= \left(\nabla \wedge \underline{h}\right) . \underline{d}S$$

We see that the circulation of any vector $\underline{h}$ around an infinitesimally small square is the component of curl $\underline{h}$ normal to the surface times the area of the square. As before, for a macroscopic loop,

$$\oint_{\Gamma} \underline{h} . \underline{d}\ell = \int_{S} \nabla \wedge \underline{h} . \underline{dS} \tag{1.36}$$

This is **Stoke's Theorem.**