Chapter 6. Magnetostatic Fields in Matter

6.1. Magnetization

Any macroscopic object consists of many atoms or molecules, each having electric charges in motion. With each electron in an atom or molecule we can associate a tiny magnetic dipole moment (due to its spin). Ordinarily, the individual dipoles cancel each other because of the random orientation of their direction. However, when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the material becomes magnetized. The state of magnetic polarization of a material is described by the parameter M which is called the **magnetization** of the material and is defined as

$$M = \text{magnetic dipole moment per unit volume}$$

In some material the magnetization is parallel to $\overline{B}$. These materials are called paramagnetic. In other materials the magnetization is opposite to $\overline{B}$. These materials are called diamagnetic. A third group of materials, also called Ferro magnetic materials, retain a substantial magnetization indefinitely after the external field has been removed.

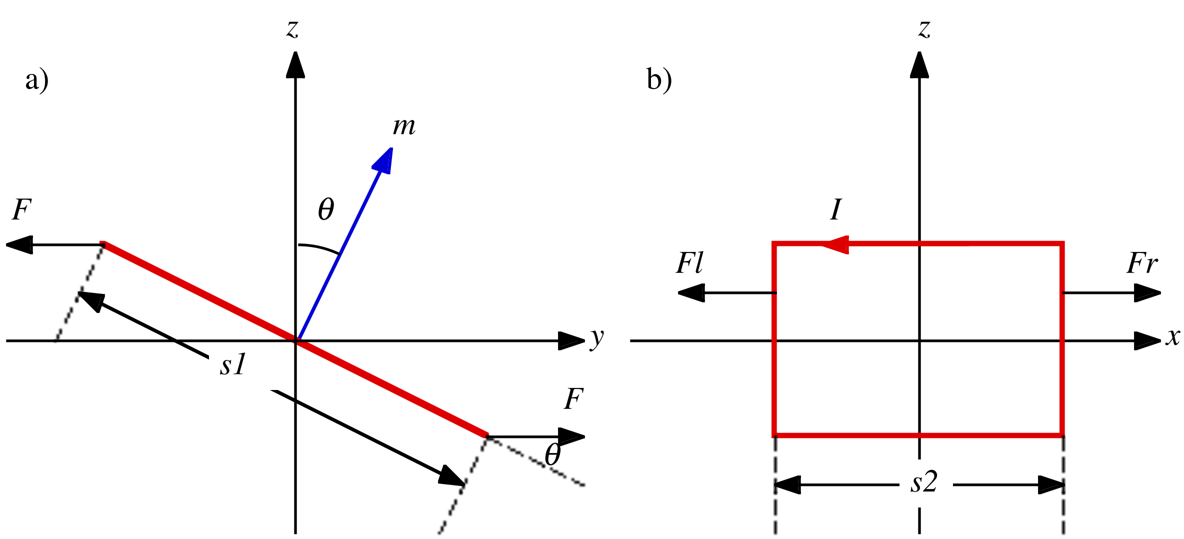

Figure 6.1. Force on a rectangular current loop.

6.1.1. Paramagnetism

Consider a rectangular current loop, with sides s_1 and s_2, located in a uniform magnetic field, pointing along the z axis. The magnetic dipole moment of the current loop makes an angle θ with the z axis (see Figure 6.1a). The magnetic forces on the left and right sides of the current loop have the same magnitude but point in opposite directions (see Figure 6.1b). The net force acting on the left and right side of the current loop is therefore equal to zero. The force on the top and bottom part of the current loop (see Figure 6.1a) also have the same magnitude and point in opposite directions. However since these forces are not collinear, the corresponding torque is not equal to zero. The torque generated by magnetic forces acting on the top and the bottom of the current loop is equal to

$$\overline{N} = \sum \overline{r} \times \overline{F} = \frac{s_1}{2} F \sin\theta \, \hat{i} + \frac{s_1}{2} F \sin\theta \, \hat{i} = s_1 F \sin\theta \, \hat{i}$$

The magnitude of the force F is equal to

$$F = I \left| \int_{Line} d\bar{l} \times \overline{B} \right| = I s_2 B$$

Therefore, the torque on the current loop is equal to

$$\overline{N} = s_1 s_2 I B \sin\theta \, \hat{i} = m B \sin\theta \, \hat{i} = \overline{m} \times \overline{B}$$

where $\overline{m}$ is the magnetic dipole moment of the current loop. As a result of the torque on the current loop, it will rotate until its dipole moment is aligned with that of the external magnetic field.

In atoms we can associate a dipole moment with each electron (spin). An external magnetic field will line up the dipole moment of the individual electrons (where not excluded by the Pauli principle). The induced magnetization is therefore parallel to the direction of the external magnetic field. It is this mechanism that is responsible for paramagnetism.

In a uniform magnetic field the net force on any current loop is equal to zero:

$$\overline{F} = I \oint \left[d\bar{l} \times \overline{B} \right] = I \left[\oint d\bar{l} \right] \times \overline{B} = 0$$

since the line integral of $d\bar{l}$ is equal to zero around any closed loop.

If the magnetic field is non-uniform then, in general, there will be a net force on the current loop. Consider an infinitesimal small current square of side ε, located in the yz plane and with a current flowing in a counter-clockwise direction (see Figure 6.2). The force acting on the current loop is the vector sum of the forces acting on each side:

$$
\begin{aligned}
\overline{F} &= I\left[\int_{Side\,1} d\bar{l}\times\overline{B} + \int_{Side\,2} d\bar{l}\times\overline{B} + \int_{Side\,3} d\bar{l}\times\overline{B} + \int_{Side\,4} d\bar{l}\times\overline{B}\right] = \\
&= I\left[\int_0^{\varepsilon}\left(\hat{j}\times\overline{B}(0,y,0)\right)dy + \int_0^{\varepsilon}\left(\hat{k}\times\overline{B}(0,\varepsilon,z)\right)dz + \int_{\varepsilon}^{0}\left(\hat{j}\times\overline{B}(0,y,\varepsilon)\right)dy + \int_{\varepsilon}^{0}\left(\hat{k}\times\overline{B}(0,0,z)\right)dz\right] = \\
&= I\left[\int_0^{\varepsilon}\left(\hat{j}\times\left[\overline{B}(0,y,0)-\overline{B}(0,y,\varepsilon)\right]\right)dy + \int_0^{\varepsilon}\left(\hat{k}\times\left[\overline{B}(0,\varepsilon,z)-\overline{B}(0,0,z)\right]\right)dz\right] = \\
&= I\left[-\int_0^{\varepsilon}\left(\hat{j}\times\varepsilon\left.\frac{\partial\overline{B}}{\partial z}\right|_{(0,y,0)}\right)dy + \int_0^{\varepsilon}\left(\hat{k}\times\varepsilon\left.\frac{\partial\overline{B}}{\partial y}\right|_{(0,0,z)}\right)dz\right]
\end{aligned}
$$

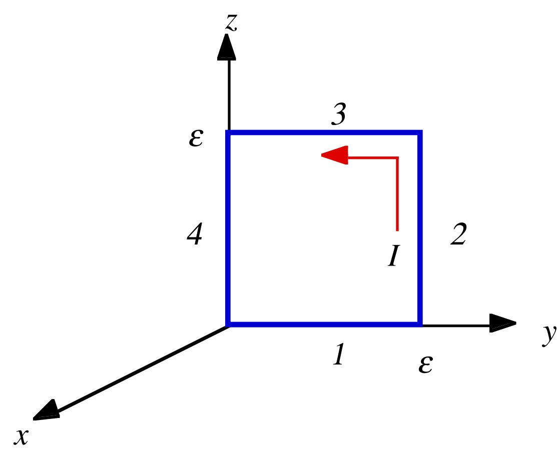

Figure 6.2. Infinitesimal square current loop.

In this derivation we have used a first-order Taylor expansion of $\overline{B}$:

$$\overline{B}(0,\varepsilon,z) = \overline{B}(0,0,z) + \varepsilon\left.\frac{\partial\overline{B}}{\partial y}\right|_{(0,0,z)}$$

and

$$\overline{B}(0,y,\varepsilon) = \overline{B}(0,y,0) + \varepsilon\left.\frac{\partial\overline{B}}{\partial z}\right|_{(0,y,0)}$$

Assuming that the current loop is so small that the derivatives of $\overline{B}$ are constant over the boundaries of the loop we can evaluate the integrals and obtain for the total force:

$$\overline{F} = I\varepsilon^2\left[\hat{k}\times\frac{\partial\overline{B}}{\partial y} - \hat{j}\times\frac{\partial\overline{B}}{\partial z}\right] = m\left[\left(\frac{\partial B_x}{\partial y}\hat{j} - \frac{\partial B_y}{\partial y}\hat{i}\right) - \left(\frac{\partial B_z}{\partial z}\hat{i} - \frac{\partial B_x}{\partial z}\hat{k}\right)\right] = m\left[\frac{\partial B_x}{\partial x}\hat{i} + \frac{\partial B_x}{\partial y}\hat{j} + \frac{\partial B_x}{\partial z}\hat{k}\right]$$

where m is the magnetic dipole moment of the current loop. In this derivation we have used the fact that the divergence of $\overline{B}$ is equal to zero for any magnetic field and this requires that

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y} - \frac{\partial B_z}{\partial z}$$

The magnetic dipole moment m of the current loop is equal to

$$\overline{m} = m\hat{i}$$

Therefore, the equation for the force acting on the current loop can be rewritten in terms of $\overline{m}$:

$$\overline{F} = \frac{\partial}{\partial x}(mB_x)\hat{i} + \frac{\partial}{\partial y}(mB_x)\hat{j} + \frac{\partial}{\partial z}(mB_x)\hat{k} = \overline{\nabla}(\overline{m}\bullet\overline{B})$$

Any current loop can be build up of infinitesimal current loops and therefore

$$\overline{F} = \overline{\nabla}(\overline{m}\bullet\overline{B})$$

for any current loop.

Example: Problem 6.1.

a) Calculate the torque exerted on the square loop shown in Figure 6.3 due to the circular loop (assume r is much larger than a or s).

b) If the square loop is free to rotate, what will its equilibrium orientation be?

a) The dipole moment of the current loop is equal to

$$\overline{m} = \pi a^2 I\,\hat{k}$$

where we have defined the z axis to be the direction of the dipole. The magnetic field at the position of the square loop, assuming that $r \gg a$, will be a dipole field with $\theta = 90°$:

$$\overline{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\hat{\theta} = \frac{\mu_0}{4\pi}\frac{\pi a^2 I}{r^3}\hat{\theta} = -\frac{\mu_0}{4}\frac{a^2}{r^3}I\,\hat{k}$$

The dipole moment of the square loop is equal to

$$\overline{m}_{square} = s^2 I\,\hat{i}$$

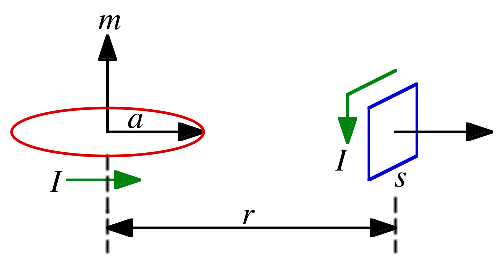

Figure 6.3. Problem 6.1.

Here we have assumed that the x axis coincides with the line connecting the center of the current circle and the center of the current square. The torque on the square loop is equal to

$$\overline{N} = \overline{m}_{square} \times \overline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ s^2 I & 0 & 0 \\ 0 & 0 & -\frac{\mu_0}{4}\frac{a^2}{r^3} I \end{vmatrix} = \frac{\mu_0}{4}\frac{a^2 s^2}{r^3} I^2 \hat{j}$$

b) Suppose the dipole moment of the square loop is equal to

$$\overline{m}_{square} = m_x \hat{i} + m_y \hat{j} + m_z \hat{k}$$

The torque on this dipole is equal to

$$\overline{N} = \overline{m}_{square} \times \overline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ m_x & m_y & m_z \\ 0 & 0 & -\frac{\mu_0}{4}\frac{a^2}{r^3} I \end{vmatrix} = -\frac{\mu_0}{4}\frac{a^2}{r^3} I\, m_y\, \hat{i} + \frac{\mu_0}{4}\frac{a^2}{r^3} I\, m_x\, \hat{j}$$

In the equilibrium position, the torque on the current loop must be equal to zero. This therefore requires that

$$m_x = m_y = 0$$

Thus, in the equilibrium position the dipole will have its dipole moment directed along the z axis. The energy of a magnetic dipole in a magnetic field is equal to

$$U = -\overline{m} \bullet \overline{B}$$

The system will minimize its energy if the dipole moment and the magnetic field are parallel. Since the magnetic field at the position of the square loop is pointing down, the equilibrium position of the current loop will be with its magnetic dipole moment pointing down (along the negative z axis).

6.1.2. Diamagnetism

Consider a very classical picture of a Hydrogen atom consisting of an electron revolving in a circular orbit of radius r around a nucleus (see Figure 6.4). Suppose that the velocity of the electron is equal to v. Since the velocity of the electron is very high, the revolving electron looks like a steady current of magnitude

$$I = \frac{e}{T} = \frac{ev}{2\pi r}$$

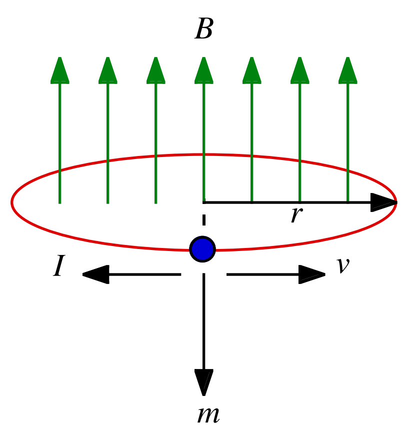

Figure 6.4. Electron in orbit.

The direction of the current is in a direction opposite to that of the electron. The dipole moment of this current is equal to

$$\overline{m} = -\frac{1}{2} evr\,\hat{k}$$

If the atom is placed in a magnetic field, it will be subject to a torque. However, it is very difficult to tilt the entire orbit. Instead the electron will try to reduce its torque by changing its velocity. With no magnetic field present, the velocity of the electron can be obtained by requiring that the centripetal force is sustained by just the electric force:

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m_e\frac{v^2}{r}$$

In a magnetic field, the centripetal force will be sustained by both the electric and the magnetic field:

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} + ev'B = m_e\frac{v'^2}{r}$$

Here we have assumed that the magnetic field is pointing along the positive z axis (in a direction opposite to the direction of the magnetic dipole moment). We have also assumed that the size of the orbit (r) does not change when the magnetic field is applied. Combining the last two equations we obtain

$$m_e\frac{v^2}{r} + ev'B = m_e\frac{v'^2}{r}$$

or

$$ev'B = \frac{m_e}{r}\left(v'^2 - v^2\right) = \frac{m_e}{r}(v'+v)(v'-v)$$

Assuming that the change in the velocity is small we can use the following approximations:

$$v' \cong v$$

and

$$v' - v \cong \Delta v$$

Therefore,

$$evB = 2\frac{m_e}{r}v\Delta v$$

This equation shows that the presence of the magnetic field will increase the speed of the electron. An increase in the velocity of the electron will increase the magnitude of the dipole moment of the revolving electron. The change in $\overline{m}$ is opposite to the direction of $\overline{B}$. If the electron would have been orbiting the other way, it would have been slowed down by the magnetic field. Again the change in the dipole moment is opposite to the direction of $\overline{B}$.

In the presence of an external magnetic field the dipole moment of each orbit will be slightly modified, and all these changes are anti-parallel to the external magnetic field. This is the magnetism that is responsible for diamagnetism. Diamagnetism is present in all materials, but is in general much weaker than paramagnetism. It can therefore only be observed in those materials where paramagnetism is not present.

6.2. The Field of a Magnetized Object

Consider a magnetized material with magnetization $\overline{M}$. The associated vector potential $\overline{A}$ is equal to

$$\overline{A} = \frac{\mu_0}{4\pi}\int \frac{\overline{M} \times \Delta\hat{r}}{\Delta r^2} d\tau$$

Following the same procedure used in Chapter 4 to calculate the electrostatic potential of a polarized material, we obtain for $\overline{A}$:

$$\overline{A} = \frac{\mu_0}{4\pi}\int_{Surface} \frac{\overline{\nabla} \times \overline{M}}{\Delta r} d\tau + \frac{\mu_0}{4\pi}\oint_{Surface} \frac{\overline{M} \times d\overline{a}}{\Delta r} = \frac{\mu_0}{4\pi}\int_{Surface} \frac{\overline{J}_b}{\Delta r} d\tau + \frac{\mu_0}{4\pi}\oint_{Surface} \frac{\overline{K}_b}{\Delta r} da$$

where $\overline{J}_b = \overline{\nabla} \times \overline{M}$ is the bound volume current and $\overline{K}_b = \overline{M} \times \hat{n}$ is the bound surface current. If the material has a uniform magnetization then the bound volume current is zero. The field produced by a magnetized object is equal to the field produced by the bound currents.

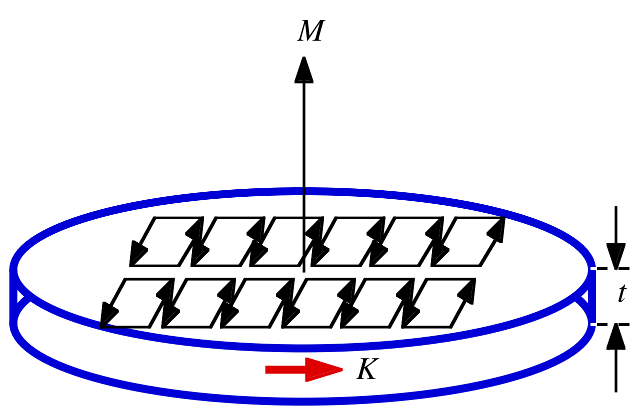

Figure 6.5. Bound surface current.

Consider a uniformly magnetized thin slab of material of thickness t. The material can be cut up into tiny current loops (see Figure 6.5). If each current loop has an area a then the dipole moment due to a surface current I is equal to

$$\overline{m}_I = Ia\hat{k}$$

The volume of the current loop is at and therefore its dipole moment must be equal to

$$\overline{m}_I = Mat\hat{k}$$

This requires that the surface current of the current loop is equal to

$$I = Mt$$

Since the magnetization is uniform, the current in each of the current loops will be constant and flowing in the same direction. Therefore, all volume currents cancel, and the only current remaining will be a surface current, flowing on the surface of the material. The current flowing on the surface of the material will be equal to the current in each of the current loops. Therefore, the current density on the surface is equal to

$$K = \frac{I}{t} = M$$

In vector notation:

$$\overline{K} = \overline{M} \times \hat{n}$$

This equation is also consistent with the fact that there is no current flowing on the top and bottom surfaces (where $\overline{M} \times \hat{n} = 0$).

Example: Problem 6.7

An infinitely long circular cylinder carries a uniform magnetization $\overline{M}$ parallel to its axis. Find the magnetic field (due to $\overline{M}$) inside and outside the cylinder.

Consider a coordinate system S in which the z axis coincides with the axis of the cylinder. The magnetization of the material is equal to

$$\overline{M} = M\hat{k}$$

Since the material is uniformly magnetized, its bound volume current is equal to zero. The bound surface current is equal to

$$\overline{K}_b = \overline{M} \times \hat{n} = \overline{M} \times \hat{r} = M\hat{\phi}$$

This current distribution is identical to the current distribution in an infinitely long solenoid. The magnetic field outside an infinitely long solenoid is equal to zero (see Example 9, Chapter 5 of Griffiths), and therefore also the field outside the magnetized cylinder will be equal to zero. The magnetic field inside an infinitely long solenoid can be calculated easily using Ampere's law (see Example 9, Chapter 5 of Griffiths). It is equal to

$$\overline{B} = \mu_0 K_b \hat{k} = \mu_0 M \hat{k}$$

6.3. The Auxiliary Field *H*

The magnetic field in a system containing magnetized materials and free currents can be obtained by calculating the field produced by the total current $\overline{J}$ where

$$\overline{J} = \overline{J}_{free} + \overline{J}_{bound}$$

This approach is very similar to the approach taken in electrostatics where the total electric field produced by a system containing dielectric materials is equal to the electric field produced by a charge distribution σ where

$$\sigma = \sigma_{free} + \sigma_{bound}$$

To calculate the magnetic field produced by a system containing magnetized materials we have to use the following form of Ampere's law:

$$\frac{1}{\mu_0}\left(\overline{\nabla} \times \overline{B}\right) = \overline{J}_{free} + \overline{J}_{bound} = \overline{J}_{free} + \overline{\nabla} \times \overline{M}$$

This equation can be rewritten as

$$\overline{\nabla} \times \left(\frac{1}{\mu_0}\overline{B} - \overline{M}\right) = \overline{J}_{free}$$

The quantity in parenthesis is called the *H*-field

$$\overline{H} = \frac{1}{\mu_0}\overline{B} - \overline{M}$$

$\overline{H}$ plays a role in magnetostatics analogous to $\overline{D}$ in electrostatics. Ampere's law in terms of $\overline{H}$ reads

$$\overline{\nabla} \times \overline{H} = \bar{J}_{free}$$

and

$$\oint \overline{H} \bullet dl = I_{\text{free,intercepted}}$$

Ampere's law for $\overline{H}$ tells us that the curl of $\overline{H}$ is equal to the free current density. However, a knowledge of the free current density is not sufficient to determine $\overline{H}$. The Helmholtz theorem shows that besides knowing the curl of a vector function, we also need to know the divergence of that vector function before it is uniquely defined. Although the divergence of $\overline{B}$ is zero for any magnetic field (and therefore Ampere's law for $\overline{B}$ defines $\overline{B}$ uniquely) the divergence of $\overline{H}$ is not necessarily zero:

$$\overline{\nabla} \bullet \overline{H} = \overline{\nabla} \bullet \left(\frac{1}{\mu_0} \overline{B} - \overline{M} \right) = \frac{1}{\mu_0} \overline{\nabla} \bullet \overline{B} - \overline{\nabla} \bullet \overline{M} = -\overline{\nabla} \bullet \overline{M}$$

Therefore, only for those systems where $\overline{\nabla} \bullet \overline{M} = 0$ can we use Ampere's law for $\overline{H}$ directly to calculate $\overline{H}$. The divergence of $\overline{H}$ will be zero only for systems with cylindrical, plane, solenoidal, or toroidal symmetry.

The $\overline{H}$ field is a quantity that is used in the laboratory more often that the $\overline{B}$ field. This is a result of the dependence of $\overline{H}$ on only the free currents (which are easy to control). The $\overline{B}$ field depends both on the free and on the bound currents, and thus requires a detailed knowledge of the magnetic properties of the materials used. In electrostatics, the electric field can be obtained immediately from the potential difference (which is easy to control). The electric displacement $\overline{D}$ depends only on the free charge distribution, but in most cases a direct measurement of the free charge distribution is very difficult to carry out. Therefore, in electrostatics the electric field is in most cases a more useful parameter then the electric displacement $\overline{D}$.

Example: Problem 6.12

An infinitely long cylinder, of radius *R*, carries a "frozen-in" magnetization, parallel to the axis,

$$\overline{M} = kr\,\hat{k}$$

where *k* is a constant and *r* is the distance from the axis (there is no free current anywhere). Find the magnetic field inside and outside the cylinder by two different methods:

a) Locate all the bound currents, and calculate the field they produce.

b) Use Ampere's law to find $\overline{H}$, and then get $\overline{B}$.

a) The magnetization of the material is directed along the z axis and is equal to

$$\overline{M} = kr\,\hat{k}$$

The bound volume current is equal to

$$\overline{J}_b = \overline{\nabla} \times \overline{M} = \frac{1}{r}\frac{\partial M_z}{\partial \phi}\hat{r} - \frac{\partial M_z}{\partial r}\hat{\phi} = -k\hat{\phi}$$

The bound surface current is equal to

$$\overline{K}_b = \overline{M} \times \hat{n}\Big|_{r=R} = kR\,\overline{k} \times \hat{r} = kR\,\hat{\phi}$$

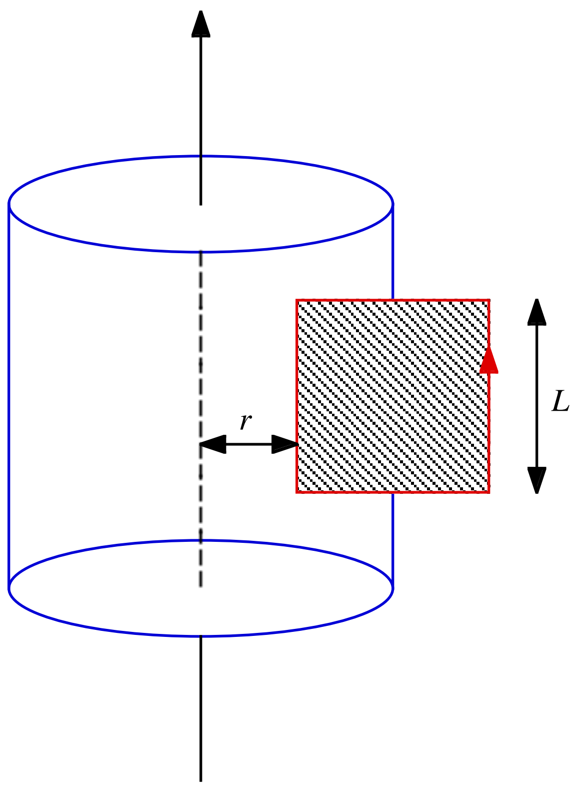

Figure 6.6. Problem 6.12.

The bound currents produce a solenoidal field. The field outside the cylinder will be equal to zero and the field inside the cylinder will be directed along the z axis. Its magnitude can be obtained using Ampere's law. Consider the Amperian loop shown in Figure 6.6. The line integral of $\overline{B}$ along the Amperian loop is equal to

$$\oint \overline{B} \bullet d\overline{l} = -BL$$

The current intercepted by the Amperian loop is equal to

$$I_{encl} = -K_b L + \int_r^R J_b L dr = -kLR + \int_r^R kL dr = -kLR + kL(R-r) = -kLr$$

Ampere's law can now be used to calculate the magnetic field:

$$\overline{B} = \frac{\mu_0 I_{encl}}{-L}\hat{k} = \mu_0 kr\,\hat{k}$$

b) The divergence of $\overline{M}$ is equal to zero. Therefore, Ampere's law uniquely defines $\overline{H}$. The $\overline{H}$ field is pointing in the z direction. Using Ampere's law, in terms of the $\overline{H}$ field, we immediately conclude that for the Amperian loop shown in Figure 6.6

$$\oint \overline{H} \bullet d\overline{l} = HL = I_{\text{free,intercepted}} = 0$$

since there is no free current This can only be true if $\overline{H} = 0$. This implies that

$$\overline{H} = \frac{1}{\mu_0}\overline{B} - \overline{M} = 0$$

Therefore, the magnetic field $\overline{B}$ is equal to

$$\overline{B} = \mu_0 \overline{M}$$

In the region outside the cylinder the magnetization is equal to zero and therefore the magnetic field is equal to

$$\overline{B} = 0$$

In the region inside the cylinder the magnetization is equal to

$$\overline{M} = kr\,\hat{k}$$

and therefore the magnetic field is equal to

$$\overline{B} = \mu_0 kr\,\hat{k}$$

which is identical to the result obtained in part a).

Example: Problem 6.14

Suppose the field inside a large piece of material is $\overline{B}_0$, and the corresponding $\overline{H}$ field is equal to

$$\overline{H}_0 = \frac{1}{\mu_0}\overline{B}_0 - \overline{M}$$

a) A small spherical cavity is hollowed out of the material. Find the $\overline{B}$ field at the center of the cavity in terms of $\overline{B}_0$ and $\overline{M}$. Also find the $\overline{H}$ field at the center of the cavity in terms of $\overline{H}_0$ and $\overline{M}$.
b) Do the same for a long needle-shaped cavity running parallel to $\overline{M}$.
c) Do the same for a thin wafer-shaped cavity perpendicular to $\overline{M}$.
Assume the cavities are small enough so that $\overline{M}$, $\overline{B}_0$, and $\overline{H}_0$ are essentially constant.

a) The field in the spherical cavity is the superposition of the field $\overline{B}_0$ and the field produced by a sphere with magnetization $-\overline{M}$. The bound volume current in the sphere is equal to zero (uniform magnetization). The bound surface current is equal to

$$\overline{K}_b = \left(-\overline{M}\right)\times\hat{n} = -M\sin\theta\,\hat{\phi}$$

Here we have assumed that the magnetization of the sphere is directed along the z axis. Now consider a uniformly charged sphere, rotating with an angular velocity ω around the z axis. The system carries a surface current equal to

$$\overline{K} = \sigma\overline{v} = \sigma\omega R\sin\theta\,\hat{\phi}$$

Comparing these two equations for the surface current we conclude that

$$M = -\sigma\omega R$$

In Example 11 of Chapter 5 the magnetic field produced by a uniformly charged, rotating sphere was calculated. The magnetic field inside the sphere was found to be uniform and equal to

$$\overline{B} = \frac{2}{3}\mu_0\sigma\omega R\hat{k}$$

But since $M = -\sigma\omega R$ we can rewrite this expression as

$$\overline{B} = -\frac{2}{3}\mu_0 M\,\hat{k} = -\frac{2}{3}\mu_0\overline{M}$$

The magnetic field inside the spherical cavity is therefore equal to

$$\overline{B}_{cavity} = \overline{B}_0 + \overline{B}_{sphere} = \overline{B}_0 - \frac{2}{3}\mu_0\overline{M}$$

The corresponding $\overline{H}$ field is equal to

$$\overline{H}_{cavity} = \frac{1}{\mu_0}\overline{B}_{cavity} - \overline{M}_{cavity} = \frac{1}{\mu_0}\overline{B}_{cavity} = \frac{1}{\mu_0}\overline{B}_0 - \frac{2}{3}\overline{M} = \overline{H}_0 + \frac{1}{3}\overline{M}$$

Here we have used the fact that $\overline{M}_{cavity} = 0$ since no materials are present there.

b) The magnetic field inside the needle-shaped cavity is equal to the vector sum of the field $\overline{B}_0$ and the field produced by a needle-shaped cylindrical piece of material with magnetization $-\overline{M}$. The field inside a needle-shaped cylinder of magnetization $-\overline{M}$ is approximately equal to the field inside an infinitely long solenoid. This field was calculated in Problem 6.7, and for a cylinder with a uniform magnetization $-\overline{M}$ it is equal to

$$\overline{B} = -\mu_0\overline{M}$$

The magnetic field inside the needle-shaped cavity is thus equal to

$$\overline{B}_{cavity} = \overline{B}_0 + \overline{B}_{cylinder} = \overline{B}_0 - \mu_0\overline{M}$$

The corresponding $\overline{H}$ field is equal to

$$\overline{H}_{cavity} = \frac{1}{\mu_0}\overline{B}_{cavity} - \overline{M}_{cavity} = \frac{1}{\mu_0}\overline{B}_{cavity} = \frac{1}{\mu_0}\overline{B}_0 - \overline{M} = \overline{H}_0$$

c) The magnetic field in the center of a thin wafer-shaped cavity is equal to the vector sum of $\overline{B}_0$ and the magnetic field inside a waver-shaped material with magnetization $-\overline{M}$. Since the thickness of the wafer approaches zero, the total surface current on the material approaches zero, and consequently the magnetic field inside the waver approaches zero. Therefore, the magnetic field inside the cavity will be equal to

$$\overline{B}_{cavity} = \overline{B}_0 + \overline{B}_{wave} = \overline{B}_0$$

The corresponding $\overline{H}$ field is equal to

$$\overline{H}_{cavity} = \frac{1}{\mu_0}\overline{B}_{cavity} - \overline{M}_{cavity} = \frac{1}{\mu_0}\overline{B}_{cavity} = \frac{1}{\mu_0}\overline{B}_0 = \overline{H}_0 + \overline{M}$$

6.4. Linear Media

In paramagnetic and diamagnetic materials, the magnetization is maintained by the external magnetic field. The magnetization disappears when the field is removed. Most paramagnetic and diamagnetic materials are linear; that is their magnetization is proportional to the $\overline{H}$ field:

$$\overline{M} = \chi_m \overline{H}$$

The constant of proportionality χ_m is called the magnetic susceptibility of the material. In vacuum the magnetic susceptibility is zero. In a linear medium, there is linear relation between the magnetic field and the $\overline{H}$ field:

$$\overline{B} = \mu_0\left(\overline{H} + \overline{M}\right) = \mu_0\left(1 + \chi_m\right)\overline{H} = \mu\overline{H}$$

where μ is called the permeability of the material. The permeability of free space is equal to μ_0.

The linear relation between $\overline{H}$ and $\overline{B}$ does not automatically imply that the divergence of $\overline{H}$ is zero. The divergence of $\overline{H}$ will only be equal to zero inside a linear material, but will be non-zero at the interface between two materials of different permeability. Consider for example the interface between a linear material and vacuum (see Figure 6.7). The surface integral of $\overline{M}$ across the surface of the Gaussian pillbox shown in Figure 6.7 is definitely not equal to zero. According to the divergence theorem the surface integral of $\overline{M}$ is equal to the volume integral of $\overline{\nabla} \bullet \overline{M}$:

$$\oint_{Surface} \overline{M} \bullet d\overline{a} = \int_{Volume} \left(\overline{\nabla} \bullet \overline{M}\right) d\tau$$

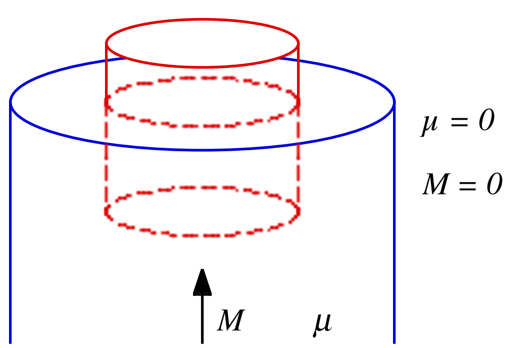

Figure 6.7. Interface of linear materials.

Therefore, if the surface integral of $\overline{M}$ is not equal to zero, the divergence of $\overline{M}$ can not be zero everywhere.

Example: Example 6.3

An infinite solenoid (N turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

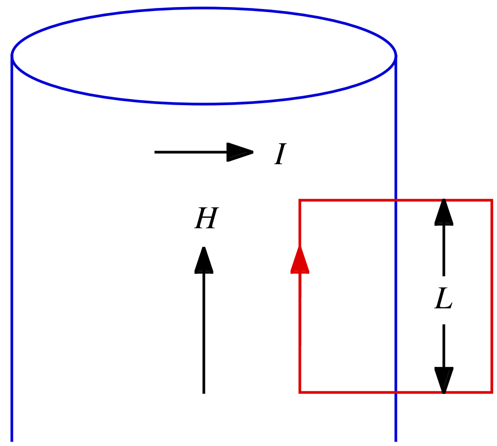

Figure 6.8. Example 6.3.

Because of the symmetry of the problem, the divergence of $\overline{H}$ will be equal to zero, everywhere. Therefore, the $\overline{H}$ field can be obtained directly from Ampere's law. Consider the Amperian loop shown in Figure 6.8. The line integral of $\overline{H}$ around the loop is equal to

$$\oint_{Line} \overline{H} \bullet d\bar{l} = HL$$

where the line integral is evaluate in the direction shown in Figure 6.8, and it is assumed that the $\overline{H}$ field is directed along the z axis. The free current intercepted by the Amperian loop is equal to

$$I_{\text{free,intercepted}} = NIL$$

Ampere's law for the $\overline{H}$ field immediately shows that

$$\overline{H} = NI\,\hat{k}$$

The magnetic field inside the solenoid is equal to

$$\overline{B} = \mu_0(1+\chi_m)\overline{H} = \mu_0(1+\chi_m)NI\,\hat{k}$$

The magnetization of the material is equal to

$$\overline{M} = \chi_m \overline{H} = \chi_m NI\,\hat{k}$$

and is uniform. Therefore, there will be no bound volume currents in the material. The bound surface current is equal to

$$\overline{K}_b = \overline{M} \times \hat{n} = \chi_m \left(\overline{H} \times \hat{n}\right) = \chi_m NI\,\hat{\phi}$$

This last equation shows that the bound surface current flows in the same direction (paramagnetic materials) or in an opposite direction (diamagnetic materials) as the free current.

Example: Problem 6.18

A sphere of linear magnetic material is placed in an originally uniform magnetic field $\overline{B}_0$. Find the new field inside the sphere.

This problem can be solved using a method similar to the method used in example 7 of Griffiths (Chapter 4). The external field $\overline{B}_0$ will magnetize the sphere:

$$\overline{M}_0 = \chi_m \overline{H}_0 = \frac{\chi_m}{\mu_0\left(1 + \chi_m\right)} \overline{B}_0$$

This magnetization will produce a uniform magnetic field inside the sphere (see Example 6.1 of Griffiths, Chapter 6):

$$\overline{B}_1 = \frac{2}{3}\mu_0 \overline{M}_0 = \frac{2\chi_m}{3\left(1 + \chi_m\right)} \overline{B}_0$$

This additional magnetic field magnetizes the sphere by an additional amount:

$$\overline{M}_1 = \frac{\chi_m}{\mu_0\left(1 + \chi_m\right)} \overline{B}_1 = \frac{2\chi_m^{\ 2}}{3\mu_0\left(1 + \chi_m\right)^2} \overline{B}_0$$

This additional magnetization produces an additional magnetic field inside the sphere:

$$\overline{B}_2 = \frac{2}{3}\mu_0 \overline{M}_1 = \left(\frac{2\chi_m}{3\left(1 + \chi_m\right)}\right)^2 \overline{B}_0$$

The total magnetic field inside the sphere is therefore equal to

$$\overline{B} = \sum_{n=0}^{\infty} \left(\frac{2\chi_m}{3(1+\chi_m)} \right)^n \overline{B}_0 = \frac{1}{1 - \frac{2\chi_m}{3(1+\chi_m)}} \overline{B}_0 = \frac{3(1+\chi_m)}{3+\chi_m} \overline{B}_0$$

When can check the consistency of this answer by calculating the magnetization of the sphere:

$$\overline{M} = \frac{\chi_m}{\mu_0(1+\chi_m)} \overline{B} = \frac{\chi_m}{\mu_0(1+\chi_m)} \frac{3(1+\chi_m)}{3+\chi_m} \overline{B}_0 = \frac{3\chi_m}{\mu_0(3+\chi_m)} \overline{B}_0$$

The magnetic field inside the sphere due to the magnetization $\overline{M}$ is equal to

$$\overline{B}_M = \frac{2}{3} \mu_0 \overline{M} = \frac{2}{3} \mu_0 \frac{3\chi_m}{\mu_0(3+\chi_m)} \overline{B}_0 = \frac{2\chi_m}{3+\chi_m} \overline{B}_0$$

The total magnetic field inside the sphere is therefore equal to

$$\overline{B}_{total} = \overline{B}_0 + \overline{B}_M = \left(1 + \frac{2\chi_m}{3+\chi_m}\right) \overline{B}_0 = \frac{3+3\chi_m}{3+\chi_m} \overline{B}_0$$

which is consistent with our assumption.

6.5. Nonlinear Media

The best known nonlinear media are the ferromagnetic materials. Ferromagnetic materials do not require external fields to sustain their magnetization (therefore, the magnetization definitely depends in a nonlinear way on the field). The magnetization in ferromagnetic materials involves the alignment of the dipole moments associated with the spin of unpaired electrons. The difference between ferromagnetic materials and paramagnetic materials is that in ferromagnetic materials the interaction between nearby dipoles makes them want to point in the same direction, even when the magnetic field is removed. However, the alignment occurs in relative small patches, called domains. When a ferromagnetic material is not located in a magnetic field, the dipole moments of the various domains are not aligned, and the material as a whole is not magnetized. When the ferromagnetic material is put into a magnetic field, the boundaries of the domain parallel to the field will increase at the expense of neighboring boundaries. If the field is strong enough, one domain takes over the entirely, and the ferromagnetic material is said to be saturated (all unpaired electrons are aligned and therefore the magnetization reaches a maximum value). The magnetic susceptibility of ferromagnetic materials is around 10^3 (roughly eighth orders of magnitude larger than the susceptibility of paramagnetic materials). When the magnetic field is removed some magnetization remains (and we have created a permanent magnet). For any ferromagnetic material, the magnetization depends not only on the applied magnetic field but also on the magnetization history. The alignment of dipoles in a ferromagnet can be destroyed by random thermal motion. The destruction of the alignment occurs at a precise temperature (called the Curie point). When a ferromagnetic material is heated above its Curie temperature it becomes paramagnetic.